# Robust Comparative Statics in Large Static Games

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*Abstract*— **We provide general comparative static results for large finite and infinite-dimensional aggregative games. In aggregative games, each player's payoff depends on her own actions and an aggregate of the actions of all the players (for example, the average of the actions among the players). In large games, players take these aggregates as given. We derive comparative static results for large aggregative games, showing both how equilibrium aggregates and the behavior of each player change in response to various different types of changes in parameters. Our results can also be interpreted as comparative statics of $\epsilon$-equilibria in games in which there is a large but finite number of players, who still take their impact on aggregates into account in choosing their strategies. We illustrate how these results can be applied easily using two examples: (1) large single or multi-dimensional contests; (2) large beauty contests where each player's strategy is a probability distribution.**

## I. Introduction

In aggregative games, each player's payoff depends on her own actions and some aggregate of all players' actions. A well known example of an aggregative game is the Cournot model of oligopoly, where each firm's profits depend on its own quantity and total quantity supplied to the market. More generally, the aggregate could be any mapping from the players' action profile to a real number or a vector. Several commonly-studied games, including the majority of the models of competition (Cournot and Bertrand with or without product differentiation), models of (patent) races, models of contests, and models of public good provision, can be cast as aggregative games. A large aggregative game is one in which each individual has infinitesimal impact on the aggregates and thus takes their behavior as given.

In this paper, we provide a simple general framework for comparative static analysis in large aggregative games. Comparative statics show how the structure of equilibria changes when parameters affecting the payoffs change. There are few general comparative static results in games. [1], [2] and [3] provide such results for supermodular games (or games with strategic complements), where each player's payoff is supermodular in all other players' strategies (as well as her own strategy), meaning that the payoff of a player increases more in her own strategy when others choose "greater" strategies. [4] provide general comparative static results for aggregative games that satisfy various concavity and local solvability assumptions.[1] In particular, define the backward reply mapping as the correspondence that gives the (best response) strategies of players that are compatible with a given value of the aggregate. The key local solvability assumption in [4] is that this backward reply mapping is invertible. In this paper, we prove results similar to those found in [4] without assuming local solvability. We also generalize the framework by allowing for infinite as well as finite-dimensional strategy spaces.

More specifically, define a "positive shock" to be a change in parameters that increases the best response of a player for a given value of the aggregate (in particular, increasing the greatest and least selection from the best response correspondence). We show that a positive shock to a subset of players always increases the value of the equilibrium aggregate. Under additional assumptions, we also show the impact of such shocks on each player's equilibrium strategies. Results of this level of generality are obtained thanks to the assumption that each player takes the aggregate(s) as given. We also show that if the number of players is indeed large, taking the aggregate(s) as given is in fact an $\epsilon$-equilibrium, and thus our results can be interpreted as approximate comparative statics for more general games.

We next provide an example of a large aggregative game, which we will later use to illustrate how our main comparative static results can be applied. To show that these results hold with infinite as well as finite-dimensional strategies, we provide a game in which strategies correspond to probability distributions.

*Example 1:* **(Beauty Contests)** *Consider a simplified version of the "beauty contest game" first proposed by John Maynard Keynes in analogy to a stock market, where each trader would like to guess other traders' guesses.*[2] *There is a continuum of players represented by the unit interval* $[0,1]$. *Player* $i \in [0,1]$ *receives an independently drawn private signal* $s_i \in S$ *and makes a prediction* $x_i(s_i) \in S$. *So a strategy for player* $i$ *is a mapping* $x_i : S \to S$ *that gives the public prediction as a function of the private signal. Let us define the average prediction given a specific realization of the private signals* $(s_i : i \in [0,1])$ *as:*

$$G(x) = \int_{[0,1]} x_i(s_i) di.$$

*We discuss the exact interpretation of this integral below. For now, it suffices to say that with the appropriate definition of the integral, the independence of the private signals ensures that* $G(x)$ *is a degenerate random variable or simply a real number ("the average prediction"). Agents care*

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[1] They also provide general comparative static results for aggregative games with strategic substitutes, where the payoff of a player increases more in her own strategy when others choose "smaller" strategies.

[2] This is simpler than a related beauty contest game discussed in [5].

*both about making guesses close to their signal and about guessing correctly (meaning that they would like to be close to the average prediction). One specific example would be the following payoff function for each player $i$:*

$$\Pi_i(x) = -\int [\alpha(x_i(s_i)-s_i)^2+(1-\alpha)(x_i(s_i)-G(x))^2]d\Gamma_i(s_i)$$

*where we have integrated over the distribution $\Gamma_i(s_i)$ of the private signal. This corresponds to a weighted quadratic loss function, with weight $\alpha$ on the gap between the prediction and private signal, and weight $1-\alpha$ on the gap between the prediction and the average prediction. Note that in this formulation we are using the fact that $G(x)$ is degenerate/a real number. Each player maximizes this function choosing a strategy $x_i : S \to S$.*

Our paper is related to the small but growing literature on robust comparative statics and to the study of aggregative games. [1], [6], [7] and [3] provide a framework for deriving comparative static results in supermodular games (games with strategic complements). Our work more directly extends [8], who provides comparative static results for aggregative games with strategic substitutes under fairly restrictive conditions, and our previous work [4], which provide more general comparative static results both for aggregative games with strategic substitutes and for aggregative games that satisfy certain smoothness and regularity conditions.

The rest of the paper is organized as follows. Section II defines aggregative games and proves the existence of (Nash) equilibria under weak regularity conditions. Section III contains our main comparative static results, which focus on games with one-dimensional aggregates. Section IV studies games with multidimensional aggregates. Section V illustrates how these results can be applied using two simple examples. Section VI establishes the relationship between large games and $\epsilon$-equilibria of finite games with sufficiently many players, so that our results can also be interpreted as comparative statics for "approximate equilibria" of finite games. Section VII concludes, while the Appendix provides more details on the interpretation of integrals in this context.

## II. Large Aggregative Games

We consider large games, *i.e.*, games populated by a non-atomic measure space of agents, represented by the unit interval $\mathcal{I} = [0,1]$ with the Lebesgue measure and Borel algebra. To simplify the exposition, we also assume that there is a finite number $M \in \mathbb{N}$ of *types* of agents, with a set of agents of type $m$ denoted by $\mathcal{I}_m$ (though this assumption can be easily relaxed). Each agent of type $m \in \{1,\ldots,M\}$ has a strategy set $X_m$. We assume that each $X_m$ is a compact metric space. We write $X_{m(i)}$ when we wish to mention specifically the index of an agent as well as her type $m(i) \in \{1,\ldots,M\}$. An element in $X_{m(i)}$ (a strategy for player $i$) is denoted by $x_i$. We also define $x_{-i} = (x_j)_{j\in[0,1]\setminus\{i\}}$, $X_{-i} = \prod_{j\in[0,1]\setminus\{i\}} X_j$, $x = (x_i)_{i\in[0,1]}$ (a strategy profile) and $X = \prod_{i\in[0,1]} X_i$ (the joint strategy set).[3] Throughout this paper, product spaces are equipped with the product topology. The payoff function of a player of type $m$ is denoted $\Pi_m$, and the payoff function of player $i$ correspondingly by $\Pi_{m(i)}$. We also allow each agent $i$ to receive a private signal/shock at the beginning of the game, denoted by $s_i \in S_{m(i)}$, drawn from some distribution $\Gamma_{m(i)}$, and assume that all draws across players are independent.[4] Payoffs depend on the full strategy profile $x$ as well as an exogenous variable $t_{m(i)} \in T_{m(i)} \subset \mathbb{R}$ with respect to which we wish to do comparative statics, *i.e.*, $\Pi_{m(i)} = \Pi_{m(i)}(x, t_{m(i)})$.

*Definition 1:* A game is *aggregative* if there exists a mapping $G : X \to \mathbb{R}^N$ such that each player's payoff function can be written as:

$$\Pi_{m(i)}(x, t_i) \equiv \pi_{m(i)}(x_i, G(x), t_{m(i)}) \quad (1)$$

This definition allows for multidimensional aggregates. For our main results in Section III, we will focus on the case where $G : X \to \mathbb{R}$, so that the aggregate is one-dimensional (the results in this section apply more generally). For this case, let us define $\Omega$ as the range of $G$, *i.e.*, $\Omega \equiv \{G(x) : x \in X\} \subseteq \mathbb{R}$. The function $\pi_m : X_m \times \Omega \times T_m \to \mathbb{R}$ in the definition of an aggregative game is referred to as the *reduced payoff function*. The value of $G$ given strategies $x$, $Q = g(x)$ is called the *aggregate* and the mapping $G$ is referred to as the *aggregator*. The notation makes it clear that we make no distinction between the case where $G(x)$ is a degenerate random variable taking the value $Q$ with probability 1, and the case where $G(x)$ is a "true" real-valued function taking the value $Q$. The justification for this will become clear below. We place the following general assumptions on the aggregator $G$:

*Assumption 1:* The aggregator $G(x) = H(\mathcal{L}(x))$, where $\mathcal{L} : X \to \mathbb{R}$ is a linear and continuous operator and $H$ is a continuous function.

In what follows, it suffices to focus on the case in which $\mathcal{L} : X \to \mathbb{R}$ corresponds to an integral. A (joint) strategy $x \in X$ is *measurable* if the mapping $i \mapsto x(i)$ is measurable. Let $x$ be a measurable strategy profile and consider the integral:

$$\int_{[0,1]} x(i)di. \quad (2)$$

When $X_i$ is a subset of the reals, the previous integral has the usual meaning. In this case, we may take as aggregator $G(x) = \int_{[0,1]} x(i)di$ or $G(x) = H\left(\int_{[0,1]} x(i)di\right)$, where $H$ is a continuous function. More generally, $X_i$ may be a subset of $\mathbb{R}^N$ and if the integral is interpreted coordinatewise, we can then take a function of the type $G(x) = H(\int_{[0,1]} x(i)di)$ (with $H : \mathbb{R}^N \to \mathbb{R}$ continuous) as an aggregator. In more complicated cases, such as when each $x(i)$ is a random

[3]It is sometimes necessary to consider a more restrictive joint strategy set than the general product space. In particular, it is sometimes necessary to consider only joint strategies $x = (x_i)_{i\in[0,1]}$ that are *measurable*. In what follows, with a slight abuse of terminology, we refer to $X$ as the product space.

[4]The independence assumption can be relaxed in a variety of settings. For example, we consider a specification in the Appendix where draws are only required to be *pairwise independent* across players.

variable as was the case in the "beauty contest" game considered in the introduction, what (2) means must be defined with some care. In the Appendix, we consider a specific way of defining (2) due to [9] and the appropriate law of large numbers that ensures that (2) will be a degenerate random variable. We may then identify the integral with a real number $G(x)$ equal to the degenerate distribution's point of unit-mass. This then is one way of constructing an aggregator $G : X \to \mathbb{R}$ when strategies are random variables. But we stress that there are many possible ways to interpret (2) that agree with our overall assumptions.

Given the assumption that there is a continuum of players, it is immediate that we have a *large game*, in the sense that for all $i \in \mathcal{I}$ and $x_{-i} \in X_{-i}$, we have $G(x_i, x_{-i}) = G(\tilde{x}_i, x_{-i})$ (for all $x_i$, $\tilde{x}_i \in X_i$), since each player is "infinitesimal". Clearly, in such a large game, players will take the aggregate $Q = G(x)$ as given when choosing their optimal strategy.[5] An immediate implication is that the *best-reply correspondence* $R_m : \Omega \times T_m \to 2^{X_m}$ of player of type $m$ can be expressed as a function of the aggregate $Q$ and the parameter $t_m$:

$$R_m(Q, t_m) \equiv \{x_i \in X_{m)} : \pi_m(x_i, Q, t_m) \geq \pi_m(\tilde{x}_i, Q, t_m) \text{ for all } \tilde{x}_i \in X_{m(i)}\}, \quad (3)$$

where we are using the fact that the best-reply correspondence of all players of a given type will be identical.

We thus define a (Nash) equilibrium in pure strategies for such large games as follows:

*Definition 2:* **(Equilibrium)** A strategy profile $x^* \in X$ is an *equilibrium* (in pure strategies) if $x_i^* \in R_{m(i)}(Q^*, t_{m(i)})$ for all $i \in \mathcal{I}$ and $Q^* = G(x^*)$.

In this definition, the statement "for all $i \in \mathcal{I}$" is a shorthand "for $i \in \mathcal{I}$ almost everywhere" since with a continuum of players, deviations by measure zero subsets of the set of players is inconsequential. Throughout we use this shorthand notation to simplify the terminology.

Finally, we also impose:

*Assumption 2:* For all $i \in \mathcal{I}$, the strategy set $X_{m(i)}$ is compact, and the payoff function $\Pi_{m(i)}$ is upper semi-continuous in the player's own strategy $x_i \in X_{m(i)}$.

Assumption 2 is weaker than the standard in non-cooperative game theory because payoff functions are not assumed to be quasi-concave in own strategies. That quasi-concavity is unnecessary stems from the well known "convexifying" effect of working with a continuum of agents ( [10], [11]).

*Theorem 1:* **(Existence)** Suppose Assumptions 1 and 2 hold. Then any large aggregative game has an equilibrium (in pure strategies).

*Proof:* Since $X_m$ is compact, and $\pi_m$ is upper semi-continuous in the player's own strategy, the best-reply correspondence of any player $R_i(Q, t_i) = \arg\max_{x_i \in X_i} \pi_{m(i)}(x_i, Q, t_i)$ is non-empty valued, compact valued and upper hemi-continuous. Let $\mathcal{G}(Q, t) \equiv \{G(x) : x \text{ is measurable and } x_i \in R_i(Q, t_i) \text{ for all } i \in \mathcal{I}\}$. For fixed $t$, the correspondence $\mathcal{G}(\cdot, t) : \Omega \to 2^{\Omega}$ will be upper hemi-continuous ( [10]), and have non-empty, compact and convex values ( [12]). The existence of an equilibrium now follows directly from Kakutani's fixed point theorem. ■

[5]The converse of this statement need not be true, in the sense that even if a game is not large as defined here, one could look for an equilibrium in which players take the aggregate as given (see Acemoglu and Jensen, 2009).

## III. Comparative Statics with One-Dimensional Aggregates

Our main interest is with comparative static results, which show how the structure of the equilibrium changes as a function of exogenous parameters. To study this, we need to place a partial order on the strategy sets. From now on, we assume that each $X_m$, in addition to being a compact metric space, is a partially ordered space equipped with a closed partial order $\succeq_m$.[6] When $X_i \subseteq \mathbb{R}^N$, $\succeq_i$ will normally be taken to be the usual Euclidean order which we denote by $\geq$. When $X_i$ is a set of random variables, the first-order stochastic dominance order is often the best choice (see *e.g.* [13]). The product order defined on $X$ is denoted $\succeq$ (in other words, $\tilde{x} \succeq x \Leftrightarrow \tilde{x}_i \succeq_i x_i$ for all $i \in [0, 1]$).

For our comparative statics results we also need to assume that $G$ is an increasing function in the strategies:

*Assumption 3:* The aggregator $G$ is increasing in the sense that $G(\tilde{x}) \geq G(x)$ whenever $\tilde{x} \succeq x$.

Next we define the central notion of a *positive shock*. Recall that $R_{m(i)}(Q, t_{m(i)})$ denotes the best-reply correspondence of player $i$ where $Q$ is the aggregate and $t_{m(i)}$ the exogenous variable.

*Definition 3:* **(Positive Shocks)** A change in the parameter from $t_m^1$ to $t_m^2$, $t_m^2 > t_m^1$, is a *positive shock* to players of type $m$ if for all $Q \in \Omega$ the following holds: (i) For all $x_i \in R_m(Q, t_m^1)$ there exists $\tilde{x}_i \in R_m(Q, t_m^2)$ with $\tilde{x}_i \succeq_m x_i$, and (ii) For all $\tilde{x}_i \in R_m(Q, t_m^2)$ there exists $x_i \in R_m(Q, t_m^1)$ with $\tilde{x}_i \succeq_m x_i$.

When $R_m(Q, t)$ is single-valued, for example when the payoff function is strictly quasi-concave in the player type's own strategy, this definition reduces to: $R_m(Q, t_m^2) \succeq_m R_i(Q, t_m^1)$ whenever $t_m^2 \geq t_m^1$. This simply means that the function $R_m(Q, t_m)$ is nondecreasing in $t_m$. If $X_m$ is a lattice, *(i)* and *(ii)* will hold provided that $R_i(Q, \cdot)$ is increasing in the strong set set order.[7] We briefly consider this important special case in the following remark.

*Remark 2: One way to ensure that (increases) in $t_m$ will be positive shocks is to use Topkis's Monotonicity Theorem ( [2], [6]). Specifically, if $X_m$ is a lattice, and $\pi_m(x_i, Q, t_m)$ is supermodular in $x_i$ and exhibits increasing differences in $x_i$ and $t_m$, then $R_m(Q, \cdot)$ will be increasing in the strong set order (given any $Q \in \Omega$). It is straightforward to verify that*

[6]That the order $\succeq_i$ is closed means that the relation $\succeq_m = \{(x, y) \in X_m^2 : x \succeq_m y\}$ is a closed subset of $X_m^2$ (here and throughout, product sets are equipped with the product topology).

[7]Given an order $\succeq_m$ on a lattice $X_m$, the strong set order $\succeq_s$ is defined on sublattices $A, B$ of $X_m$ as follows: $A \succeq_s B \Leftrightarrow [a \vee b \in A$ and $a \wedge b \in B$ for all $a \in A$ and $b \in B]$ (here $\vee$ denotes the supremum of $a$ and $b$ in $X_m$ and $\wedge$ denotes the infimum).

*if $R_m(Q, t_m^2) \succeq_s R_i(Q, t_m^1)$ then (i) and (ii) of Definition 3 will hold. For a generalization of Topkis' theorem, see [14]. For explicit conditions in the case where strategies are random variables, see [13]. One may also establish that a shock is positive by using the implicit function theorem (which does not require $X_i$ to be a lattice, but instead requires smoothness and convexity assumptions on payoff functions and strategy sets).*

Theorem 1 guarantees the existence of an equilibrium, though not its uniqueness. Let $t = (t_m)_{m=1}^M$ denote the exogenous variables for the player types, and for a given $t$, denote the *set* of equilibria (in pure strategies) by $E(t) \subseteq X$. For every equilibrium $x^*(t) \in E(t)$ we have a well-defined equilibrium aggregate $G(x^*(t))$. It is not hard to show that $E(t)$ will be a compact set, so when $G$ is continuous (assumption 1), we may define the *smallest* and *largest* equilibrium aggregates:

$$Q_*(t) \equiv \min_{x^*(t)\in E(t)} G(x^*(t)) \text{ and } Q^*(t) \equiv \max_{x^*(t)\in E(t)} G(x^*(t)) \quad (4)$$

Our main comparative statics theorem, following next, tells us that under the previous assumptions, $Q_*(t)$ and $Q^*(t)$ are nondecreasing when the change in $t$ constitutes a positive shock, *i.e.*, a positive shock for each of the types of players whose $t_m$ is changed.

*Theorem 3:* **(Main Comparative Statics Result)** Consider a large aggregative game satisfying Assumptions 1-3. Then the smallest and largest equilibrium aggregates $Q_*(t)$ and $Q^*(t)$ are nondecreasing in a positive shock.

*Proof:* Let $\mathcal{G}(Q,t) \equiv \{G(x) : x_i \in R_i(Q,t_i) \text{ for all } i \in \mathcal{I}\}$. From the proof of Theorem 1, $\mathcal{G}(\cdot,t) : \Omega \to 2^\Omega$ is non-empty, convex and compact valued and upper hemi-continuous. It is clear that $Q^* = G(x^*)$ where $x^* \in E(t)$ if and only if $Q^* \in \mathcal{G}(Q^*,t)$. Hence there is a one-to-one correspondence between fixed points for the correspondence $\mathcal{G}(\cdot,t)$ and equilibria in the game. Since $\mathcal{G}$ is compact valued, the least and greatest selections of $\mathcal{G}$ are well-defined: $\underline{g}(Q,t) \equiv \inf \mathcal{G}(Q,t)$ and $\overline{g}(Q,t) \equiv \sup \mathcal{G}(Q,t)$. When each $R_m$ is increasing in $t_m$ in the sense of the definition of a positive shock, it can be shown that $\underline{g}(Q,t)$ and $\overline{g}(Q,t)$ will be increasing functions of $t$ (see [15]). That the least and greatest fixed points of $\mathcal{G}$ (which are precisely the smallest and largest equilibrium aggregates) will be increasing in $t$ now follows directly from Corollary 2 in [16]. ∎

*Corollary 1:* **(Individual Comparative Statics)** Consider a large aggregative game satisfying Assumptions 1-3. Consider a player type $m \in \{1,\dots,M\}$, and assume that for this type of players, $X_m$ is a lattice, and that $\pi_m(x_i,Q,t_m)$ is supermodular in $x_i$, and exhibits increasing differences in $x_i$ and $Q$. Then the strategies of the players of type $m$ associated with the smallest and the largest equilibrium aggregates will be nondecreasing in a positive shock to any subset of agents (not necessarily the ones of type $m$). If instead $\pi_m(x_i,Q,t_m)$ exhibits decreasing differences in $x_i$ and $Q$, then the strategies of the players of type $m$ associated with the smallest and largest equilibrium aggregates will be nonincreasing in a positive shock to any type of players different from $m$.

*Proof:* The first statement is a direct consequence of Topkis's Monotonicity Theorem ( [6]) since under the stated conditions, $R_m(Q,t_m)$ will be non-decreasing in $Q$ in the strong set order (and so an increase in $Q$ will increase the smallest and greatest strategies for players of type $m$, whether or not $t_m$ is changed). The second conclusion is proved similarly, now using that $R_m(Q,t_m)$ will be nonincreasing in $Q$ in the strong set order and that $t_m$ is held fixed. ∎

## IV. Comparative Statics with Multidimensional Aggregates

It is also possible to derive comparative static results with multidimensional aggregates, i.e., when $G : X \to \mathbb{R}^N$ (assuming that each coordinate function $G_n : X \to \mathbb{R}$ satisfies the above conditions). Recall that Theorem 1 and our definition of positive shocks were general enough to cover multidimensional aggregates. In what follows, we interpret $G(\tilde{x}) \geq G(x)$ in Assumption 3 to mean that each component of $G(\tilde{x})$ is greater than or equal to the corresponding component of $G(x)$. The following theorem provides an analog of Theorem 3 in the case of multidimensional aggregates.

*Theorem 4:* **(Supermodular Games Case)** Consider a large aggregative game satisfying Assumptions 1-3. In addition, assume that for each $m \in \{1,\dots,M\}$, $X_m$ is a lattice, and $\pi_m(x_i,Q,t_{m(i)})$ is supermodular in $x_i$ and exhibits increasing differences in $x_i$ and $Q$. Then the strategies of all players associated with the smallest and largest equilibrium aggregates are nondecreasing in a positive shock (and these equilibrium aggregates are also nondecreasing in a positive shock).

*Proof:* The proof follows along the same lines as the proof of Theorem 3, except that one now uses Theorem 4 in [16] instead of Corollary 2. ∎

*Remark 5: Theorem 4 is not a special case of the well-known comparative statics results for supermodular games ( [2]). In particular, our definition of a positive shock is weaker than that of Topkis (who requires that best-reply correspondences are nondecreasing as a function of the parameter in the strong set order).*

The next theorem provides a partial converse to Theorem 4 for the submodular (or strategic substitutes) case. We will focus on two-dimensional aggregates and on an increasing shock to the first coordinate of the aggregate defined as a change in parameter from $t^1$ to $t^2$ such that $G_1(x,t^2) \geq G_1(x,t^1)$ (and $G_2(x,t^2) \geq G_2(x,t^1)$).[8] Then we have:

*Theorem 6:* **(Submodular Games Case)** Consider a large aggregative game satisfying Assumptions 1-3, and suppose that the aggregate is two dimensional ($G : X \to \mathbb{R}^2$). In addition, assume that for each $m$, $X_m$ is a lattice, and $\pi_m(x_i,Q,t_m)$ is supermodular in $x_i$ and exhibits decreasing differences in $x_i$ and $Q$. Then the largest and smallest

[8] In the supermodular or the one-dimensional cases, such increasing shocks to the aggregate can be cast as positive shocks affecting all types of players and are thus covered by Theorems 3 and 4.

first coordinate of the aggregate are nondecreasing (and the corresponding smallest and largest second coordinate of the aggregate are nonincreasing) in an increasing shock to the first coordinate of the aggregate.

*Proof:* $\mathcal{G}$ is defined as in the proof of Theorem 1, except that now it is a correspondence from $\Omega \subseteq \mathbb{R}^2$ into itself where $\Omega$ is the range of $G$. Equilibria correspond to fixed points of $\mathcal{G}$: $Q_1 \in \mathcal{G}_1(Q_1, Q_2, t)$ and $Q_2 \in \mathcal{G}_2(Q_1, Q_2, t)$. The least and greatest selections from $\mathcal{G}_1$ and $\mathcal{G}_2$ (cf. the proof of Theorem 3) will be nonincreasing functions of $Q_1$ and $Q_2$ under the assumptions of the theorem. Let us next replace $Q_2$ with $-\tilde{Q}_2$, so that an equilibrium is given by $Q_1 \in \mathcal{G}_1(Q_1, -\tilde{Q}_2, t)$ and $\tilde{Q}_2 \in -\mathcal{G}_2(Q_1, -\tilde{Q}_2, t)$. Then under the assumptions of the theorem, the least and greatest selections are increasing in $(Q_1, \tilde{Q}_2)$ and in $t$. The rest of the proof now follows the proof of Theorem 4 and is omitted. ■

## V. Examples

We now briefly discuss two examples to illustrate how easily the methods developed in this paper can be applied.

### *A. Generalized Contests*

In contests, players exert costly effort in order to win a prize (race or war). The probability that a given player wins the prize is an increasing function of her effort and decreasing function of other players' efforts. Consider the following generalization of contests to a large game, with the set of players represented by the unit interval. A fraction of the players will win a prize, which is worth $V$. The cost of a player $i$ of type $m$ of exerting effort $x_i$ is $c_m(x_i)/t_m$, where $c_m$ is a continuous and strictly increasing cost function and $t_m$ is a parameter scaling the cost function. Moreover, $x_i \in [0, \bar{x}_m]$, for some $\bar{x}_m < \infty$ (this ensures compactness of strategy sets). The probability that player $i$ will be one of the winners of the prize is given by $f_m(x_i)/H\left(\int_0^1 h(x_i)\, di\right)$, where $f_m$'s, $h_m$'s and $H$ are continuous and strictly increasing functions. Therefore, the payoff of player $i$ of type $m$ is

$$\frac{f_m(x_i)}{H\left(\int_0^1 h_m(x_i)\, di\right)} V - \frac{c_m(x_i)}{t_m}.$$

A specific application might be one where $x$ corresponds to test preparation, affecting test scores and a certain fraction of the students who score relatively high in this test will get admitted to a selective school.

It can be easily verified that this game satisfies Assumptions 1-3 and that an increase in $V$ satisfies our definition of a positive shock (for all types). Theorem 3 then implies that an increase in $V$ will (weakly) increase the aggregate $Q = H\left(\int_0^1 h_m(x_i)\, di\right)$. In addition, payoff functions exhibit decreasing differences in own strategies and the aggregate. Hence from Theorem 3 and Corollary 1, a change in $t_m$ will (weakly) increase $H\left(\int_0^1 h_m(x_i)\, di\right)$ and $x_i$ for (almost all) players of type $m$, and (weakly) decrease $x_i$ for (almost all) players of type $m' \neq m$.

These results are interesting in part because in finite contests the effects of a positive shock to a set of players is in general indeterminate, and may increase or decrease the effort of other players (see, for example, [4] for a characterization).

We can also use a further generalization of contests to show how our multidimensional aggregate results can be applied in this case. In particular, suppose that there are two types of effort, each corresponding to a separate contest. For example, one contest can again correspond to some educational competition, while the other represents competition in another realm, for example, in sports. Thus the strategy of player $i$ of type $m$ is now $\left(x_i^1, x_i^2\right) \in \left[0, \bar{x}_m^1\right] \times \left[0, \bar{x}_m^2\right]$, and the cost function is $c_m\left(x_i^1, x_i^2\right)/t_m$, where $c_m$ is continuous and strictly increasing in both of its arguments and supermodular. This latter assumption imposes the natural property that the marginal cost of exerting a particular type of effort is increasing in the amount of the other effort. The probability of winning each type of prize, worth $V^1$ and $V^2$, is only a function of the distribution of effort for that specific contest. This implies that the payoff function is a direct generalization of the one above:

$$\frac{f_m^1\left(x_i^1\right)}{H^1\left(\int_0^1 h_m^1\left(x_i^1\right) di\right)} V^1 + \frac{f_m^2\left(x_i^2\right)}{H_2\left(\int_0^1 h_m^2\left(x_i^2\right) di\right)} V^2 - \frac{c_m\left(x_i^1, x_i^2\right)}{t_m},$$

where again the $f$, $h$ and $H$'s are continuous and strictly increasing functions. Let us define $\tilde{x}_i^2 = -x_i^2$, so that the payoff function can be written in a way that satisfies the conditions of Theorem 6. In particular,

$$\frac{f_m^1\left(x_i^1\right)}{H^1\left(\int_0^1 h_m^1\left(x_i^1\right) di\right)} V^1 + \frac{f_m^2\left(-\tilde{x}_i^2\right)}{H_2\left(\int_0^1 h_m^2\left(-\tilde{x}_i^2\right) di\right)} V^2 - \frac{c_m\left(x_i^1, -\tilde{x}_i^2\right)}{t_m},$$

where now the payoff of each player is supermodular in her strategy vector $\left(x_i^1, \tilde{x}_i^2\right)$. We can then follow the steps in the proof of Theorem 6 and define the two aggregates as $Q_1 = H^1\left(\int_0^1 h_m^1\left(x_i^1\right) di\right)$ and $\tilde{Q}_2 = -H_2\left(\int_0^1 h_m^2\left(-\tilde{x}_i^2\right) di\right)$. It then follows that an increase in $V^1$ will raise $H^1\left(\int_0^1 h_m^1\left(x_i^1\right) di\right)$ and reduce $H_2\left(\int_0^1 h_m^2\left(-\tilde{x}_i^2\right) di\right)$. This comparative static is intuitive: an increase in the rewards to exerting effort in sports will raise the total amount of effort in sports, but crowd out effort in education. Once again, this type of unambiguous comparative static results would not be valid in finite contests.

### *B. Beauty Contests*

Next consider the example of beauty contests discussed in the Introduction. Consider a slight generalization where the payoff function of player $i$ of type $m$ is more generally given by (again integrating over private signals $s_i$)

$$\Pi_m(x_i) = \int [h_{1,m}(x_i(s_i), s_i, t_m) - h_{2,m}(|x_i(s_i) - G(x)|)] d\Gamma_i(s_i),$$

where $h_{1,m}$ and $h_{2,m}$ are continuous for each $m$, and $h_{2,m}$ is strictly increasing, and the aggregator is

$$G(x) = H\left(\int_{[0,1]} x_i(s_i) di\right),$$

where $H$ is a continuous and strictly increasing function. Clearly the payoff function in the Introduction is a special case. It can be verified that this is a large aggregative game that satisfies all of our assumptions. Consider an increase in $t_m$ and suppose that $h_{1,m}$ exhibits increasing differences in $x_i(s_i)$ and $t_m$ (for given $s_i$). Then this is a positive shock to type $m$. It is then an immediate implication of Theorem 3 and Corollary 1 that both $G(x)$ will increase and the function $x_i(\cdot)$ for (almost) all players of any type will "shift up". Intuitively, the increase in $t_m$ implies that given their private signals players of type $m$ would like to choose a higher $x_i$. As a result, given other players' strategies, the aggregate $G(x)$ will increase and because each player would like to be close to the aggregate prediction, they would also increase their predictions. An application of this result is that when a subset of forecasters become more aggressive and all forecasters dislike deviating too much from the average, all forecasters will forecast more aggressively.

## VI. Epsilon Equilibria

We next establish the relationship between the equilibria of large games and $\epsilon$-equilibria of corresponding finite games. We then use this relationship to reinterpret our results as comparative statics for "approximate equilibria" of finite games.

Consider a game similar to the one defined in Section II but with only a *finite* number $N \in \mathbb{N}$ of each type of players.[9] The set of indices of the players of type $m$ is denoted by $I_N(m)$. A strategy profiles is denoted by $x = (x_i)_{i=1}^{MN}$. The strategy set of an agent of type $m$ is as in Section II. The aggregator $G$ is taken to be the average across the players:

$$G_N(x) = \frac{1}{NM} \sum_{m=1}^{M} \sum_{i \in I(m)} x_i. \tag{5}$$

Note that since, in general, strategies are random variables, $G_N(x)$ will be a random variable as well. The payoff function of a player $i$ of type $m$ is defined as

$$\Pi_m(x) = \pi_m(x_i, G_N(x)) =$$
$$\pi_m(x_i, \frac{1}{NM} x_i + \frac{1}{NM} \sum_{m=1}^{M} \sum_{j \in I^i(m)} x_j),$$

where $I^i(m) = I(m)$ if $i \notin I(m)$ and $I^i(m) = I(m)\backslash\{i\}$ if $i \in I(m)$ and we have omitted exogenous variables since they are not relevant for the results in this section. A (pure strategy Nash) *equilibrium* for the game with $N$ players of each type, denoted $x^{*,N}$, is a strategy profile such that $x_i^{*,N} \in R_{m(i)}(\frac{1}{NM}\sum_{m=1}^{M}\sum_{j \in I^i(m)} x_j^*)$ for all $i$ where $R_m$ is the best-reply correspondence of a player of type $m$. We also define an *$\epsilon$-equilibrium* $x^{*,N}$, where $\epsilon > 0$, as a strategy profile such that $x_i^{*,N} \in \mathcal{B}_\epsilon[R_m(\frac{1}{NM}\sum_{m=1}^{M}\sum_{j \in I^i(m)} x_j^{*,N})]$ for all $i$, where $\mathcal{B}_\epsilon(Z)$ denotes the closed epsilon ball around the set $Z$. Note that best-replies depend on the "average of all other players strategies" which in general is a random variable. Under the general assumptions of Section II (all of which are in effect throughout this section), $R_m$ will be upper hemi-continuous, in particular $R_m(\frac{1}{NM}\sum_{m=1}^{M}\sum_{j \in I^i(m)} x_j)$ will be a closed set. In addition, let us suppose that all payoff functions are strictly quasi-concave. This ensures that each $R_m$ is a continuous function (a single-valued upper hemi-continuous correspondence is automatically continuous).

Then for any $i$:

$$\lim_{N\to\infty} G_N = \lim_{N\to\infty} \frac{1}{NM} \sum_{m=1}^{M} \sum_{j \in I^i(m)} x_j, \tag{6}$$

and if the integral across stochastic variables are defined as explained in the Appendix, after a trivial reindexing of the players to lie within the unit interval we have that:[10]

$$\lim_{N\to\infty} G_N = \int_{[0,1]} x_i di \tag{7}$$

From the previous construction one gets the following result as a direct consequence of the continuity of the best-reply correspondences:

*Theorem 7:* **($\epsilon$-Equilibria)** Fix a sequence of finite games with increasing number of players of each type (and the same strategy sets and payoff functions), and let $\{x^{*,N}\}_{N=1}^{\infty}$ denote any sequence of equilibria of these games and $x^*$ be an accumulation point of this sequence. Then $x^*$ will be an equilibrium in the large (continuum) game of Section II with strategy sets and payoff functions identical to those of the finite game. Conversely, let $x^*$ be any equilibrium in the corresponding large game. Then for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ (where $N$ generally depends on $\epsilon$) such that $x^*$ will be an $\epsilon$-equilibrium in the finite player game with $N$ players of each type.

*Proof:* Omitted to save space. ■

This theorem establishes, under strict quasi-concavity of payoff functions, the connection between equilibria large games and $\epsilon$-equilibria of finite games with sufficiently many players. Given this connection, all of our results on comparative statics in large games can be interpreted as comparative statics results for $\epsilon$-equilibria of finite games with sufficiently many players. Notably, in such finite games

[9] This assumption is adopted for notational simplicity and all of the results in this section generalize with minimal modifications if instead there is a finite number $N_m$ of players of type $m$.

[10] Specifically, this can be done by letting a game with $N$ agents of each type be indexed by $\{0, \frac{1}{NM}, \frac{2}{NM}, \ldots, 1\}$ rather than by $\{1, 2, \ldots, NM\}$ as done above.

players do *not* take aggregates as given. Thus in the beauty contest as well as the generalized contest examples, we can obtain unambiguous comparative statics results (for $\epsilon$-equilibria, where $\epsilon$ may be chosen to be arbitrarily small) as long as the number of players is sufficiently large. This is important since as already mentioned in Section V such unambiguous comparative statics results need not hold in finite and "small" games (and the same observation clearly applies to $\epsilon$-equilibria of "small" games).

## VII. Conclusion

This paper studied comparative statics of equilibria in large finite and infinite-dimensional aggregative games. In aggregative games, each player's payoff depends on her own actions and an aggregate of the actions of all the players (for example, the average of the actions among the players). In large games, players take these aggregates as given. We derived comparative static results for large aggregative games. We then illustrated how they can be applied in a very straightforward manner using two examples: (1) large single or multi-dimensional contests; (2) large beauty contests where each player's strategy is a probability distribution. These examples highlight how the fact that we are dealing with large games enables much stronger comparative static results than would be possible in games with a finite number of players. We also establish conditions under which our results can be interpreted as comparative statics for "approximate" equilibria of finite games (which sufficiently many players), where players do not take these aggregate as given.

## Appendix: Interpretation of Integrals

Here we present a concrete way to interpret (2) when the integral is taken over random variables and show how leads leads to a suitable definition of an aggregator $G : X \to \mathbb{R}$. Following [9], take:

$$\int_{[0,1]} x(i)di \equiv \lim_{n\to\infty} \sum_{i=1}^{n} x(t_i)(t_i - t_{i-1}) \qquad (8)$$

where the convergence is in $L^2$-norm, and as $n \to \infty$, the lengths of the subdivision $0 = t_1 < t_2 < \ldots < t_n = 1$ tends to zero. Given this interpretation, $\int_{[0,1]} x(i)di$ will itself be a random variable, but when the $x(i)$'s satisfy assumptions of some appropriate law of large numbers as discussed below, the distribution will be degenerate. We may then identify the integral with a real number $G(x)$ equal to the degenerate distribution's point of unit-mass.[11]

Next, let us turn to the needed law of large numbers (LLN). There are many formulations of the LLN in the literature that will suit our purpose. The following formulation (due to [17]) is general enough to include all of this paper's applications:[12]

*Theorem 8:* ( **[17]**) Let $x_1, x_2, x_3, \ldots$ be a sequence of pairwise independent random variables such that $Var[x_i] \leq b$ for some $b \geq 0$ and all $i \geq 1$. Let:

$$A_n \equiv \frac{x_1 + \ldots + x_n}{n} \text{ , and } \mu_n \equiv \mathbb{E}[A_n]$$

Then for every $\epsilon > 0$, $\lim_{n\to\infty} \Pr\{|A_n - \mu_n| \geq \epsilon\} = 1$. In particular, if the random variables are (uniformly) bounded almost surely and $\lim_{n\to\infty} \mu_n = \mu$, then,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} x_i$$

exists in $L^2$-norm and is a degenerate random variable that takes the value $\mu$ with probability one.

[11] See [9] for further details of this approach, including a discussion of its suitability as a description of the limit of an increasing sequence of games with a finite number of players.

[12] Chebyshev's weak law of large numbers can be found in any textbook on probability theory. When each $X_i$ is bounded (uniformly), the variances are also uniformly bounded, in particular their expected values exist. So the weak law of large numbers applies. The convergence notion in Chebyshev's weak law of large numbers is convergence in probability which, however, is equivalent to $L^2$-norm convergence when the random variables are bounded almost surely.

## References


[1] D. M. Topkis, "Equilibrium points in nonzero-sum $n$-person submodular games," *SIAM Journal of Control and Optimization*, vol. 17, pp. 773–787, 1979.

[2] ——, *Supermodularity and Complementarity*. New Jersey: Princeton University Press, 1998.

[3] X. Vives, "Nash equilibrium with strategic complementarities," *Journal of Mathematical Economics*, vol. 19, pp. 305–321, 1990.

[4] D. Acemoglu and M. K. Jensen, "Aggregate comparative statics," 2009, CEPR Discussion Paper No. 7254, London.

[5] D. Martimort and L. Stole, "Aggregate representations of aggregative games," 2010, mimeo.

[6] D. M. Topkis, "Minimizing a submodular function on a lattice," *Operations Research*, vol. 26, pp. 305–321, 1978.

[7] P. Milgrom and J. Roberts, "Rationalizability, learning, and equilibrium in games with strategic complementarities," *Econometrica*, vol. 58, pp. 1255–1277, 1990.

[8] L. Corchón, "Comparative statics for aggregative games. the strong concavity case," *Mathematical Social Sciences*, vol. 28, pp. 151–165, 1994.

[9] H. Uhlig, "A law of large numbers for large economies," *Economic Theory*, vol. 8, pp. 41–50, 1996.

[10] R. J. Aumann, "An elementary proof that integration preserves upper semi-continuity," *Journal of Mathematical Economics*, vol. 3, pp. 15–18, 1976.

[11] D. Schmeidler, "Equilibrium points of nonatomic games," *Journal of Statistical Physics*, vol. 7, pp. 295–300, 1973.

[12] R. J. Aumann, "Integrals of set-valued functions," *Journal of Mathematical Analysis and Applications*, vol. 12, pp. 1–12, 1965.

[13] H. A. Hopenhay and E. Prescott, "Stochastic monotonicity and stationary distributions for dynamic economies," *Econometrica*, vol. 60, pp. 1387–1406, 1992.

[14] P. Milgrom and C. Shannon, "Monotone comparative statics," *Econometrica*, vol. 62, pp. 157–180, 1994.

[15] D. Acemoglu and M. K. Jensen, "Robust comparative statics in large dynamic economies," 2010, mimeo.

[16] P. Milgrom and J. Roberts, "Comparing equilibria," *American Economic Review*, vol. 84, pp. 441–459, 1994.

[17] P. L. Chebyshev, "Des valuers moyennes (on mean values)," *Journal de Math'ematics Pures et Appliqu'ees*, vol. 12, pp. 177–184, 1867.