Appendix A

Proofs, Chapter 2
On the following pages the "long-run frontiers theorem" is proved. A loose statement of the result, and a description of the implications, is given in section 2.2. A precisely statement will be given in a moment.

The basic set-up is as in the previous sections: Discrete time, time-stationary CES utility, and time-stationary technology. To simplify the exposition we will assume that the economy consists of $M \in \mathbb{N}$ capital goods and a single primary resource, labor, which is in constant supply at all dates. It is furthermore assumed that the first capital good is also used for consumption.\footnote{The proof is easily generalized to an infinite number of capital goods and primary resources. Not all of these are necessarily produced at all dates (this is an important feature of models where the \textquote{engine} of growth is introduction of new goods). Moreover, the consumption good could also be a \textquote{pure consumption good}, i.e., a good which is not used for production. Exogenously growing factors can be accounted for if the supply of these grow at a constant rate over time. This last situation is different from a technical perspective since the production set will not be time-stationary. Finally, it creates no difficulties to change the present proof from discrete to continuous time.}

A vector of outputs at date $t$ is denoted $y_t = (y^1_t, \ldots, y^M_t)$, and $k_t = (k^1_t, \ldots, k^M_t)$ denotes a vector of inputs. Labor demand (supply) at date $t$ is denoted $L^d_t$ ($L^s_t$). As usual $c_t$ denotes consumption at date $t$. The $M+1$ markets clear at date $t$ provided that:
\begin{align*}
    c_t + k^1_t &= y^1_t \\
    k^n_t &= y^n_t, \; n = 2, \ldots, M \\
    L^s_t &= L^d_t
\end{align*}
(A.1)

The first market is referred to as the consumption-investment market, the last $M$ markets as the \textit{factor markets}. A production sequence $(y_t, k_{t-1}, L^d_t)_{t=1}^\infty$ is called \textit{balanced} if all reproducible goods expand at a constant rate, $g > 0$:
\begin{align*}
    y_t &= (1+g)^t y_0, \; t = 0, 1, 2, \ldots \\
    k_{t-1} &= (1+g)^t k_{t-1}, \; t = 1, 2, \ldots
\end{align*}
(A.2)

Likewise a consumption sequence is \textit{balanced} if it grows at the constant rate $g > 0$; $c_t = (1+g)^t c_0, \; t = 0, 1, 2, \ldots$. If the consumption and production sequences are both balanced and markets clear, (A.1), the allocation is called a \textit{balanced growth path} (BGP). A balanced growth path is summarized by $(y_0, k_{t-1}, L^s_t, c_0, g)$. $(p_0, p_1, p_2, \ldots)$, where $p_t = (p^1_t, \ldots, p^M_t) \in \mathbb{R}^M_{++}$, denotes a sequence of prices. We make no assumptions on how such a sequence arises. It may be taken as given by the firms (competitive markets), chosen by the firms (imperfect competition), or some mixture of the two (e.g. a perfectly competitive final good sector and monopolistic producers of capital goods [39]). The price of the primary resource (wages), is denoted $w_t$ at date $t$. A sequence of prices \textit{induces an overall rate of interest}, $r$, iff $p_t = (\frac{1}{1+r})^t p_0, \; \text{some} \; p_0 > 0$. The timing of the economy is as in the previous sections: At
date 0 the consumer holds an exogenously given vector of resources \( y_0 = (y_0^1, \ldots, y_0^M) \) which is sold at the market price \( p_0 \). At date 0 the consumer also buys the consumption good \( c_0 \), and firms invest \( k_0 \). However, production does not take place in period 0. Thus labor demand will equal zero at \( t = 0 \). It follows that the (aggregate) consumer’s budget equation reads:

\[
\sum_{t=0}^{\infty} p_t^c c_t = p_0 y_0 + \sum_{t=1}^{\infty} w_t L^c
\]  

(A.3)

The objective of the consumer is to maximize discounted utility \((2.1),\) section 2.2) given (A.3).

Due to time-stationarity, the production sector may be represented by a correspondence, \( S : \mathbb{R}^{2M+1}_+ \rightarrow \mathbb{R}^{2M+1}_+ \cup \{\emptyset\} \):

\[
(y_t, k_{t-1}, L^d_t) = (y_t^1, \ldots, y_t^M, k_{t-1}^1, \ldots, k_{t-1}^M, L^d_t) \in S(p_t, p_{t-1}, w_t) \quad (A.4)
\]

For any given set-up on the production side (for example perfect competition or Bertrand monopolistic competition) \( S \) represents the ‘black box’ which describes the prices and allocations that are in agreement with firms’ aggregate behavior. More precisely, the price-wage vector \((p_t, p_{t-1}, w_t)\) and the aggregate choice \((y_t, k_{t-1}, L^d_t)\) is a possible outcome on the production side if and only if \((y_t, k_{t-1}, L^d_t) \in S(p_t, p_{t-1}, w_t)\) (so it is the graph of \( S \) which is interesting).

A balanced price sequence, \( p_t = (\frac{1}{1+r})^t p_0 \), is said to it support the balanced production sequence \((y_t, k_{t-1}, L^d_t, g)\) if there exists a wage sequence \((w_1, w_2, w_3, \ldots)\) such that:

\[
((1+g)^t y_0, (1+g)^t k_{t-1}, L^d_t) \in S(p_t, p_{t-1}, w_t), \quad t = 1, 2, 3, \ldots \quad (A.5)
\]

If a BGP is (i) supported at the production side by a balanced price sequence with interest rate \( r \), and (ii) The consumer maximizes utility holding the resources \( y_0 \) at date 0, and taking the price and wage sequences as given, the economy is said to be in a balanced growth equilibrium (BGE). The following assumption states that if a balanced production sequence is price supported in the previous sense, it yields zero aggregate profit at all dates.

**Assumption A.1** Let \((p_0, p_1, p_2, \ldots) \) induce the overall interest rate \( r > 0 \), i.e., \( p_t = (\frac{1}{1+r})^t p_0, \ p_0 > 0 \). Assume that a balanced production sequence \((y_0, k_{t-1}, L^d_t, g)\) is supported by \((p_0, p_1, p_2, \ldots) \) at some wage sequence \((w_1, w_2, w_3, \ldots)\). Then \((y_t, k_{t-1}, L^d_t) \in S(p_t, p_{t-1}, w_t)\) implies that \( p_t y_t - p_{t-1} k_{t-1} - w_t L^d_t = 0 \), at all \( t = 1, 2, 3, \ldots \).

**Remark A.1:** Since the statement in assumption A.1 concerns the (very) long run, it is a natural consequence of free entry into the production sector. Note that assumption A.1 makes no statements about profits outside such a
balanced sequence (i.e., in equilibrium, at the adjustment paths towards it). Clearly, the assumption does not exclude for example the situation where an intermediate sector is monopolistic and earns positive profit which is used to cover expenses from buying patents ([39]).

Assumption A.1 turns out to have strong implications for the set of wage sequences that can support a balanced production sequence if the price sequence is balanced.

Lemma A.1 Assume that the conditions of assumption A.1 are satisfied. Then the wage sequence, \((w_1, w_2, w_3, \ldots)\), must be of the form:

\[
w_t = \left(\frac{1 + g}{1 + r}\right)^tw_0, \quad t = 1, 2, 3, \ldots
\]

(A.6)

where \(w_0 = \frac{\log(1 + r) - \log(1 + g)}{L^d} > 0\).

Proof: For the balanced production sequence A.1 implies that:

\[
\left(\frac{1}{1 + r}\right)^tp_0(1 + g)^ty_0 - \left(\frac{1}{1 + r}\right)^{t-1}(1 + g)^tk_{t-1} - w_tL^d = 0, \quad \text{all } t
\]

Canceling out terms and rearranging yields:

\[
p_0y_0 - (1 + r)p_0k_{t-1} = \left(\frac{1 + r}{1 + g}\right)^tw_tL^d
\]

Clearly the left-hand-side is constant over time. This implies that also the right-hand-side is constant. Consequently \((w_t)_{t=0}^{\infty}\) is of the form: \(w_t = \left(\frac{1 + g}{1 + r}\right)^tw_0\), where \(w_0\) is as stated in the lemma. Q.E.D.

Note as a corollary to the proof that:

\[
p_0y_0 - (1 + r)p_0k_{t-1} - w_0L^d = 0
\]

(A.7)

Below we shall make reference to this fact by saying that the production sector earns zero long-run profit.

Definition A.1 The production frontier is defined by a correspondence \(P : \mathbb{R}_{++} \Rightarrow \mathbb{R}_{++} \cup \{0\}\) such that \((r, g) \text{ 'lay at the production frontier'}\) if and only if \(r \in P(g)\).

For some rate of growth \(g > 0\), \(P(g) \subset \mathbb{R}_{++} \cup \{0\}\) is determined as the set consisting of all interest rates \(r > 0\) for which there exist a price vector \(p_0 > 0\) and a balanced production sequence \((y_0, k_{-1}, L^d, g)\) such that:

\[
((1 + g)^ty_0, (1 + g)^tk_{t-1}, L^t) \in S((\frac{1}{1 + r})^tp_0, (\frac{1}{1 + r})^{t-1}p_0, (\frac{1 + g}{1 + r})p_0\frac{1}{L^d}y_0 - (\frac{1 + r}{1 + r})k_{t-1}), \quad t = 1, 2, 3, \ldots
\]

(A.8)
and

\[(1 + g)k^m_{-1} = y^m_0, \quad m = 2, \ldots, M\]

\[L^s = L^d\]  \hspace{1cm} (A.9)

It is convenient to give a similar definition of the consumption frontier (CF) (This, of course, corresponds to the definition given in section 2.2).

**Definition A.2** The consumption frontier is defined by a single-valued or empty correspondence, \(C : R_{++} \rightarrow R_{++} \cup \{\emptyset\}\), where \(C(g) = \delta^{-1}(1 + g)^\alpha - 1\) if \(C(g) > 0\), \(C(g) = \emptyset\) otherwise. \((r, g) \text{ lay at the consumption frontier} \) if and only if \(r \in C(g)\) \((r = C(g), \text{ unless } C \text{ is empty})\).

**Remark A.2.**: \(P\) is clearly determined from the production side alone, \(C\) from the consumption side alone. When \(r \in P(g) \cap C(g)\) we say that the PF and CF intersect at \((r, g)\).

Before stating and proving the main theorem another lemma is needed (the proof is omitted, see section 2.2 for a similar - although somewhat simpler - argument):

**Lemma A.2** Let \(y_0 \gg 0\), \(L^s > 0\), and \(g > 0\) be given. If the consumer faces a balanced price sequence \(p_t = \left(\frac{1}{1+r}\right)^t p_0\), the wage sequence is as stated in lemma A.1, and \(g < r\), then the following conditions are necessary and sufficient for the balanced consumption sequence, \(c_t = (1 + g)^t c_0\), to be utility maximizing for the consumer:

\[p_0^{\dagger} c_0 = (1 - \frac{1 + g}{1 + r}) p_0 y_0 + \frac{1 + g}{1 + r} w_0 L^s\]  \hspace{1cm} (A.10)

\[r = \delta^{-1}(1 + g)^\alpha - 1\]  \hspace{1cm} (A.11)

where \(w_0\) is given by lemma A.1.

**Remark A.3.**: The case where \(g \geq r\) for some price supported and market clearing balanced production sequence, leads to non-existence of a BGE (unless the consumer is assumed to maximize utility under some 'over-taking' criterion). Thus, when the economy is described, assumptions must be placed on the firms’ 'ability to expand' (maximal \(g\)) relative to \(r\) in order to ensure the existence of a BGE. For general assumptions ensuring \(g < r\) and further discussion, the reader is referred to [?], especially assumptions 4-6 and the proof of theorems 1 and 2.

**Theorem A.3** Let the production and consumption frontiers be given by \(P\) and \(C\) as defined above. Then the following is true:
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For any pair \((r, g)\) where 'the frontiers intersect', i.e., \(r \in P(g) \cap C(g)\), there exists a balanced growth equilibrium with \(g\) as growth rate, and \(r\) as interest rate.

Conversely, any pair \((r, g)\) which is associated with a balanced growth equilibrium will satisfy \(r \in P(g) \cap C(g)\).

**Proof:** pick an arbitrary pair \((r, g)\) such that \(r \in P(g)\) and \(r = C(g)\). By definition of the PF there exist \(p_0\) and \((y_0, k_{-1})\) such that (A.8) and (A.9) are satisfied for the chosen pair \((r, g)\). Since \(r = C(g)\) equation (A.11) of lemma A.2 is satisfied. Consequently the consumer will be maximizing by choosing a balanced plan, \(c_t = (1+g)^t c_0\), if and only if (A.10) is satisfied. By lemma A.1, \(u_0\) is uniquely determined from \(r, p_0, (y_0, k_{-1})\), and \(L^d = L^s\) (factor market clearing). Hence we may use (A.10) to determine \(c_0\) 'backwards' such that utility maximizing is ensured.

We will now show that if \(c_0\) is picked as described above, the following equation is satisfied:

\[
p_0^1(y_0^n - c_0 - (1+g)k_{-1}^1) + \sum_{m=2}^{M} p_0^m(y_0^m - (1+g)k_{-1}^m) + \frac{1+g}{1+r} u_0(L^s - L^d_t) = 0
\]

(A.12)

To this end rewrite (A.12) as:

\[
p_0^1 c_0 = \left| \sum_{m=1}^{M} p_0^m(y_0^m) + \frac{1+g}{1+r} u_0 L^s \right| - (1+g) \sum_{m=1}^{M} k_{-1}^m - \frac{1+g}{1+r} u_0 L^d_t
\]

Since (A.10) holds for the chosen \(c_0\), the left-hand-side equals \(1 - \frac{1+g}{1+r} p_0 y_0 + \frac{1+g}{1+r} u_0 L^s\). Inserting this yields:

\[
0 = \frac{1+g}{1+r} \sum_{m=1}^{M} p_0^m y^m - (1+g) \sum_{m=1}^{M} k_{-1}^m - \frac{1+g}{1+r} u_0 L^d_t
\]

Now multiply by \(\frac{1+r}{1+g}\):

\[
0 = \sum_{m=1}^{M} p_0^m y^m - (1+r) \sum_{m=1}^{M} k_{-1}^m - u_0 L^d_t = p_0(y_0 - (1+r)k_{-1}) - u_0 L^s
\]

Since the firm earns zero long-run profit (A.7), this last equation is always satisfied, and the claim has been established.

As a consequence of (A.12), factor market clearing (A.9) and \(p_0^1 > 0\) implies that the consumption-investment market will automatically clear: \(y_0^n - c_0 - (1+g)k_{-1}^1 = 0\). But then the first statement in the theorem has been established: \((r, g)\) gives rise to a balanced growth equilibrium. Note that the
uniqueness of \( a_0 \) ensures that the economy could not initially 'jump' to another level of initial consumption, since this would violate the consumer's budget condition.

Since the PF and CF are clearly necessary conditions for the economy to be in a BGE the converse statement is a trivial consequence of the definitions. Q.E.D.
Appendix B

Dynamics, Chapter 2.3
Equilibria of the economy in chapter 2, section 3 satisfy the first welfare theorem, hence are Pareto optimal. Therefore the market solution will coincide with the planner’s problem which generally takes the following form:¹

\[
\max \sum_{t=0}^{\infty} \delta^t \frac{1}{1-\alpha} c_t^{\alpha} \\
\text{s.t.} \begin{cases} \\
\alpha_t = y_t^c - 0.85\lambda_t^A - 0.55\lambda_t^B \\
\lambda_t^A + 1.05\lambda_{t+1}^A - 0.85\lambda_{t+1}^B - 0.55\lambda_{t+1}^B, \quad t = 1, 2, 3, \ldots \\
1 \geq \lambda_t^A + \lambda_t^B \\
1.04\lambda_t^A + 1.07\lambda_t^B \geq \lambda_{t+1}^A + \lambda_{t+1}^B, \quad t = 1, 2, 3, \ldots \\
\alpha_t, \lambda_t^A, \lambda_t^B \geq 0, \text{ all } t \end{cases}
\]  

(B.1)

\(\lambda_t^A \geq 0\) is the scale at which the technique of type A, \(T^A = (0.85, 1, 1.5, 1.04)\), is operated at date \(t\) (similarly for \(\lambda_t^B\) where \(T^B = (0.55, 1, 1.05, 1.07)\)). The (normalized) initial stock at date 0 is \((y_0^c, y_0^b) = (y_0^c, 1)\). The objective of the planner is to find a sequence \((\lambda_t^A, \lambda_t^B)_{t=1}^{\infty}\) which maximizes the objective function in (B.1).

We will only consider the dynamics of the situation where \(\alpha = 0\), amounting to a horizontal consumption frontier. In particular, we wish to show that the BGEs associated with the counter-intuitive effect of a tax increase reported in section 3, are locally stable. We can state precisely what is meant by "local": The initial ratio of ordinary to human capital must be strictly larger than 0.85 if the (unique) BGE is that where firm A is active and the growth rate is 4%. The ratio must be strictly larger than 0.55 if the relevant BGE is that where firms of type \(B\) support a growth rate of 7%. Note that the BGE output ratio will be either \(\frac{15}{104}\) (type \(A\)) or \(\frac{106}{107}\) (type \(B\)), hence any small perturbation will comfortably place the initial output ratio above the mentioned values.²

As explained in section 2, the CF will, given a proportional tax, read:

\[
r = \frac{\delta^{-1}(1 + g)^\alpha - 1}{1 - \tau}
\]  

(B.2)

In the case where \(\alpha = 0\), this implies:

\[
1 + r = \frac{\delta^{-1} - \tau}{1 - \tau}
\]  

(B.3)

¹For the first good, the market balance equations will bind with equality due to monotonicity.

²The restriction on the initial ratio will imply that the inequalities in the planner’s problem bind with equality. This will imply that the shadow prices (multipliers) are strictly positive.
Outside the BGE, the interest rate thus determined is the implicit interest rate of the first good (which is thus constant at any interior solution). Hence, 
\[
p^*_t = \left( \frac{1}{1 + r} \right)^t p^*_0.
\]
Solving (B.1) therefore leads to the following necessary conditions (where \( p^*_t \) is the shadow price of the second good, \( p^*_t \) the shadow price of the first good, which we have just shown grows at the rate \( \frac{1}{1 + r} \)):
\[
1.5p^*_t + 1.04p^*_t \leq 0.85p^*_t + p^*_t \quad (B.4)
\]
and,
\[
1.05p^*_t + 1.07p^*_t \leq 0.55p^*_t + p^*_t \quad (B.5)
\]
Now, define \( \tilde{p}^*_t = \frac{p^*_t}{p^*_t} \) and insert \( p^*_t = \left( \frac{1}{1 + r} \right)^t \) (normalizing \( p^*_0 = 1 \)). Then the two necessary conditions may be written:
\[
\tilde{p}^*_t \leq \frac{1 + r}{1.04} [0.85 + \tilde{p}^*_t] - \frac{1.5}{1.04} \quad (B.6)
\]
respectively,
\[
\tilde{p}^*_t \leq \frac{1 + r}{1.07} [0.55 + \tilde{p}^*_t] - \frac{1.05}{1.07} \quad (B.7)
\]
Given an initial relative price \( \tilde{p}^*_0 \), (B.6)-(B.7) determine the entire price sequence \( (\tilde{p}^*_t)_{t=1}^\infty \) in a competitive equilibrium. We wish to argue that the initial relative price will always "jump" to the relative price supporting a BGP, \( \tilde{p}^* \), determined by setting \( \tilde{p}^* = \tilde{p}^*_t = \tilde{p}^*_t \) in the equations above. As shown in the paper, this will be unique whenever the CF intersects with one of the PFs vertical segments (and then \( \lambda^*_t = 0 \) all \( t \) or \( \lambda^*_t = 0 \) all \( t \), corresponding to only one of the firm types being operative).

Take some \( \tilde{p}^*_t \neq \tilde{p}^* \). Clearly, the dynamical system (B.6)-(B.7) is unstable because the coefficient to \( \tilde{p}^*_t \) is unambiguously larger than unity (the interest rate will always be larger than the maximum growth rate of the system: \( 1 + r > 1.07 \)).

Hence, if \( \tilde{p}^*_0 \neq \tilde{p}^* \), it follows that \( \tilde{p}^*_t \to +\infty \) as \( t \to \infty \) (if \( \tilde{p}^*_0 < \tilde{p}^* \), then \( \tilde{p}^*_t \to -\infty \), if \( \tilde{p}^*_0 > \tilde{p}^* \), then \( \tilde{p}^*_t \to +\infty \)). Consider now the transversality condition (TVC) associated with the planner’s problem (recall that the prices are the (strictly positive) shadow prices for a solution):
\[
\lim_{t \to \infty} p^*_t \eta^*_t + p^*_t \eta^*_t = 0 \quad (B.8)
\]
Since \( \frac{\tilde{p}^*_t}{\tilde{p}^*_t} = \frac{1}{1 + r} \frac{\tilde{p}^*_t}{\tilde{p}^*_t} \to +\infty \), it follows that for the TVC to hold: \( \eta^*_t \to 0 \) as \( t \to \infty \). But then \( \eta^*_t, c_t \to 0 \). This means, of course, that the scales \( \lambda^*_t, \lambda^*_t \) go to zero over time. This is a contradiction (this can be shown directly, or we can appeal to theorem 4.2 (chapter 4), which predicts that the consumption sequence will grow without bounds in a competitive equilibrium).