Lecture Notes, Econ 320B. Set # 3.

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1 Feasible Allocations, Pareto Optimality, and the First Welfare Theorem

Definition 1 *Consider an exchange economy with I consumers, n goods, and endowment vector* $\mathbf{e} = (\mathbf{e}^1, \ldots, \mathbf{e}^I)$. *The* **set of feasible allocations** *in this economy is given by:*

$$F(\mathbf{e}) = \{\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^I) \in \mathbb{R}_+^{In} : \sum_{i=1}^{I} \mathbf{x}^i = \sum_{i=1}^{I} \mathbf{e}^i\}$$

Note that in this definition we have implicitly assumed that each consumer's consumption set is $\mathbb{R}_+^n$ (this is because we require the allocation $\mathbf{x}$ to lie in $\mathbb{R}_+^{In}$ - which is the same as saying that each $\mathbf{x}^i$ must lie in $\mathbb{R}_+^n$).

The set of feasible allocations you should simply think of as "all possible ways to divide the economy's endowments $\mathbf{e}$ between the consumers". So if you're a planner and free to take everyone's resources and distribute them in whichever way you like; you'll be choosing some $\mathbf{x} \in F(\mathbf{e})$. In the case of two consumers and two goods $I = n = 2$, there is a neat way to depict the set of feasible allocations called an **Edgeworth box**. I'll draw one of these for you at the lectures. As you'll see, the Edgeworth box *is* the set of feasible allocations in this case.

Next we define the important notion of **Pareto efficiency** also known as **Pareto optimality**. This is definition 5.1. in JR (slightly modified because we look at utility functions here rather than preference relations).

Definition 2 *A feasible allocation* $\mathbf{x} \in F(\mathbf{e})$ *is* **Pareto optimal** *(or* **Pareto efficient***) if there is no other feasible allocation* $\mathbf{y} \in F(\mathbf{e})$*, such that:*

$$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i) \text{ for all } i = 1, \ldots, I \text{ with at least one strict inequality.} \tag{1}$$

Remark 1 *If for* $\mathbf{x} \in F(\mathbf{e})$ *there exists a* $\mathbf{y} \in F(\mathbf{e})$ *such that* (1) *holds, we say that* $\mathbf{x}$ *is* **Pareto dominated** *by* $\mathbf{y}$ *or that* $\mathbf{y}$ **Pareto dominates** $\mathbf{x}$. *Hence,* $\mathbf{x} \in F(\mathbf{e})$ *is Pareto optimal if and only if it is NOT Pareto dominated by any other feasible sequence.*

We've seen that the set of feasible allocation is all possible ways to divide an economy's resources. A way to divide the resources is thus Pareto optimal if there isn't another way to divide the resources which makes everyone at least all well off and at least one consumer strictly better off. This is a kind of minimum requirement from a social welfare perspective: There is no way we would ever want to choose an allocation that is not Pareto optimal. This does not go much further than this, though. For example, if u^1 is strictly/strongly increasing; an example of a Pareto optimal allocation is $\mathbf{x}^1 = \sum_{i=1}^{I} \mathbf{e}^i$, and $\mathbf{x}^2 = \ldots = \mathbf{x}^I = 0$, that is "give everything to the first guy and nothing to anyone else". This is Pareto optimal because *any* other feasible allocation would necessarily take something away from the first consumer who would consequently become *strictly worse* off because his utility function is strictly increasing. This example should teach you once and for all that Pareto optimality is not a "fairness" concept no matter how you define fairness (a topic we'll have much more to say about in this course).

It is possible to "draw" the set of Pareto optimal allocations in an Edgeworth box. The resulting curve which I'll (also) draw at the lectures, is called **the contract curve** (so just to repeat: contract curve=set of Pareto optimal allocations in an Edgeworth box).

Now to the first welfare theorem. This says that market economies passes the above "minimum requirement" because market equilibria are always Pareto optimal. This is Theorem 5.7. in GR.

Theorem 1 **(First Welfare Theorem)** *Consider an exchange economy* $(u^i, \mathbf{e}^i)_{i \in \mathscr{I}}$ *and assume that each* u^i *is strongly/strictly increasing on* $\mathbb{R}^n_+$ *(the consumption set). Then every Walrasian equilibrium allocation is Pareto optimal.*

We will not prove this result directly, but we will prove a result ("WEAs are in the Core theorem") which is actually more general. More about this in due time.

2 Blocking and the Core

Consider an economy with I consumers, so $\mathscr{I} = \{1, \ldots, I\}$ is the *set* of consumers. A **coalition** is a subset of $\mathscr{I}$, $S \subseteq \mathscr{I}$.

Example 1 *Let $I = 2$ so $\mathscr{I} = \{1, 2\}$. There are then three coalitions, namely $S_1 = \{1\}$, $S_2 = \{2\}$, and $S_3 = \{1, 2\}$.*

Example 2 *Let $I = 3$ so $\mathscr{I} = \{1, 2, 3\}$. There are then seven coalitions, namely $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$, $S_4 = \{1, 2\}$, $S_5 = \{1, 3\}$, $S_6 = \{2, 3\}$, and $S_7 = \{1, 2, 3\}$.*

It is clear that $S = \mathscr{I}$ is always a coalition, and it is called the **grand coalition** ("the coalition consisting of everyone in the economy"). In the two previous examples, the grand coalitions are, respectively, $\{1, 2\}$ and $\{1, 2, 3\}$. Any coalition consisting only of a single consumer that is, a coalition of the type $\{i\}$, $i \in \mathscr{I}$ is called a **singleton coalition**. For example with $\mathscr{I} = \{1, 2, 3\}$ we have the singleton coalitions $\{1\}$, $\{2\}$, and $\{3\}$ (clearly, there are always I singleton coalitions in an economy with I consumers).

Definition 3 (JR Definition 5.2) *Let $S \subseteq \mathscr{I}$ denote a coalition of consumers. We say that S* **blocks** *a feasible allocation $\mathbf{x} \in F(\mathbf{e})$ if there is an allocation $\mathbf{y} = (\mathbf{y}^1, \ldots, \mathbf{y}^I)$ such that:*

1. $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$. *[Coalition Feasibility]*
2. $u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i)$ *for all $i \in S$ with at least one strict inequality. [Coalition Pareto Dominance]*

If a feasible allocation $\mathbf{x} \in F(\mathbf{e})$ cannot be blocked by any coalition, we say that it is an **unblocked feasible allocation**.

Of course we could instead of utility functions in *2.* use preference relations and the condition would then read: $\mathbf{y}^i \succeq \mathbf{x}^i$ for all $i \in S$ with at least one strict preference.

What does it mean that a feasible allocation is blocked (by a coalition S) ? It means simply that the group of people S could "go solo" (leave the economy), and divide their resources among themselves in such a way that everyone will be at least as well of as before and someone strictly better off.[1] If you think of an economy as a country and imagine that resources had been divided in such a way that there is a blocking coalition; this country will in some sense be "unstable": A group of individuals (the blocking coalition) would be motivated to separate from the country and divide their own resources among themselves. This way of thinking leads directly to the concept of the **core**: A feasible allocation is in the core if no group would want to break away in the previous sense:

Definition 4 (JR Definition 5.3.) *The* **core** *of an exchange economy with endowments* $\mathbf{e}$*, denoted* $C(\mathbf{e})$*, is the set of all unblocked feasible allocations.*

To gain a better understanding of the core, it is helpful to look at two special kinds of coalitions: The grand coalition and the singleton coalitions.

What does it mean that a feasible allocation is *not* blocked by the grand coalition ($S = \mathscr{I}$) ? Well, if you look at the definition of a blocking coalition and put $\mathscr{I}$ in place of S you get the two conditions:

1. $\sum_{i \in \mathscr{I}} \mathbf{y}^i = \sum_{i \in \mathscr{I}} \mathbf{e}^i$. [This is feasibility: $\mathbf{x} \in F(\mathbf{e})$!]
2. $u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i)$ for all $i \in \mathscr{I}$ with at least one strict inequality. [This is the Pareto dominance]

See it ? To require that no such $\mathbf{y}$ exists leads **exactly** to the definition of Pareto optimality. So we conclude that:

Not being blocked by the grand coalition=Pareto optimality

What does it mean that a feasible allocation is not blocked by any of the singleton coalitions (by which is meant a coalition consisting of a single consumer) ? So now the coalition is $S = \{i\}$ where i is some consumer. The conditions from the blocking definition now read:

1. $\mathbf{y}^i = \mathbf{e}^i$. [The consumer consumes his initial resources...]

[1]In fact, with strictly increasing utility functions, it is then always possible to make *everyone* in the coalition strictly better off by taking a little bit from the person who's already strictly better off (just a little, she still has to be strictly better off!) and divide it among the rest.

2. $u^i(\mathbf{y}^i) > u^i(\mathbf{x}^i)$ [...and becomes strictly better of by doing so]

I guess I've already said all there is to say in the square brackets. But let's repeat: That no singleton coalition blocks a feasible allocation means that "no one consumer would be better off going solo and simply consume her initial resources".

3 All WEAs are in the Core

It is a little surprising (I think) that market economies actually lead to unblocked feasible allocations: The invisible hand always "divides" stuff in such a way that no "splinter" group of individuals would want to break away. This is the following theorem (Theorem 5.6. on p.201 in JR):

Theorem 2 * **(All WEAs are in the Core)** *Consider an exchange economy* $(u^i, \mathbf{e}^i)_{i \in \mathscr{I}}$. *If each consumer's utility function* u^i *is strictly increasing on* $\mathbb{R}^n_+$, *then every Walrasian equilibrium allocation is in the core.*

If we denote the set of WEAs by $W(\mathbf{e})$, we can write the conclusion of the previous theorem more compactly as:

$$W(\mathbf{e}) \subseteq C(\mathbf{e})$$

It will no doubt fill you with joy that we're going to prove this result at the lectures (hence the attached star). This is in the book and I won't repeat it here.

As a final remark, note that since any WEA is in the core, it is *in particular* not blocked by the grand coalition. As we have seen above, this is the same as saying that the WEA is Pareto optimal. Thus, the first welfare theorem is actually a special case (a weaker statement) than the "WEAs are in the core" theorem. Since we have proved the "WEAs in the core theorem", we have therefore also proved the first welfare theorem !