Global Stability and the "Turnpike" in Optimal Unbounded Growth Models

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Abstract

This study proves various global stability results for unbounded optimal growth models. The main theorem states that any optimal path converges into the neighborhood of a balanced growth path if future utility is sufficiently weakly discounted. The assumptions allow for non-smooth technologies, joint production, and production in independent sectors. Hence, the results form the integration of new growth and turnpike theory sought by McKenzie (1998) in his Ely lecture. The applicability of the results is exemplified by means of a number of cases from growth theory and other areas of economics.

Keywords: Optimal growth, new growth theory, homogeneous programming, turnpike, balanced growth path, global stability, von Neumann equilibrium.

JEL-classification: O41, C61, C62.

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1 Introduction

In optimal growth theory, an important questions is whether and under what conditions, steady states will be globally attractive allocations of the dynamical system describing optimal trajectories. This is the topic of *turnpike theory.* If a dynamic, representative consumer economy satisfies the conditions of the first and second welfare theorems, its equilibrium allocations coincide with the set of solutions to a social planner's problem of the type:[1]

$$\begin{array}{l} \max \sum_{t=0}^{\infty} \delta^t u(x_t) \\ \text{s.t.} \begin{cases} x_t \leq z_t - k_t \\ (-k_{t-1}, z_t) \in Y \subset \mathbb{R}_-^N \times \mathbb{R}_+^N, \text{all t} \\ x_t, z_t, k_t \in \mathbb{R}_+^N, \text{all t} \\ z_0 > 0 \textit{ given} \end{cases} \end{array} \tag{1}$$

The applicability of the general problem (1) thus includes decentralized competitive growth models, and more generally Pareto optima of possibly suboptimal market economies. Yet, the applicability is much wider including models from fertility theory, development economics, and investment planning.[2]

Typical for solutions to (1) is that dynamics is extremely complex and cannot be analyzed analytically out of steady state. Steady states, on the other hand, are simple to compute and analyze - hence, if a turnpike theorem is available, the researcher can study the long-run properties of even complex multidimensional systems with the greatest of ease.

Historically, turnpike theorems for the optimal growth model with an undiscounted objective a la Ramsey (1928) were proved during the late sixties (Atsumi (1965), Gale (1967), McKenzie (1968)). The discounted case required a more sophisticated method of proof, obtained some ten years later (Cass and Shell (1976), Scheinkman (1976)). Bewley (1982), and Yano (1984), (1985) extended the scope of the model to encompass general equilibrium models with any number of consumers, and proved turnpike theorems under various assumptions. In all of these studies the main conclusion is that equilibrium allocations will eventually be arbitrarily close to a steady state, provided that the future is sufficiently weakly discounted (*i.e.*, that the discount factor δ, is sufficiently close to unity, or in the undiscounted case, equal to it).

A fundamental feature of the results mentioned so far is that the set of feasible allocations lay in a *uniformly bounded* set. Technically put, the feasibility set is contained in l_∞, where l_∞ denotes the set of supremum bounded infinite sequences. This arises, normally, from the entry of a fixed supply of primary resources in Y, leading to the existence of some $B > 0$, such that $\parallel z \parallel \leq B$ whenever $(-k, z) \in Y$. The relevant steady states are then, of course, *stationary* allocations.[3] However, turnpike theory emerged within a rather different class of models, where the relevant

[1] This will be defined more precisely in the paper. The notation is the usual one: x_t denotes consumption, k_t (z_t) the vector of capital inputs (outputs) at date t. Y is the production set (which may depend on the supply of primary factors). u is the instant utility function, and $\delta \in (0, 1]$ the discount factor.

[2] Alvarez and Stokey (1998) study "homogeneous programs" and supply a great many examples of potential applications. Since a homogeneous program can be written in the form (1) and satisfies this paper's general assumptions, Alvarez and Stokey is an important reference for this paper.

[3] As is well known, models with factor augmenting exogenous technical progress (or increasing labor supply)

steady-states are non-stationary but grow at a constant rate with fixed relative composition: *balanced growth paths*. Thus, the first rigorous turnpike theorem due to Radner (1961) showed that all equilibrium trajectories will be attracted to a balanced growth path in a generalized von Neumann model where utility is derived from the terminal state (a "finite term" or "Samuelson" turnpike).

In the seventies the von-Neumann model faded out of the mainstream of dynamic economics which turned to the bounded optimal growth model just described (or its exogenous growth extension). Recent theoretical and empirical developments have once again turned the ties, however, directing attention toward expanding economies and balanced growth paths whose rate of growth is endogenously determined. Indeed, *overwhelming* attention has been devoted to the study of so-called new growth models, *i.e.*, models where technology is quasi-stationary but productive resources never the less capable of unbounded growth (for a good survey see Jones and Manuelli (1997)). A large body of literature (see Jones and Manuelli (1990), Rebelo (1991), Dolmas (1996), and references therein) study models of optimal unbounded growth, hence integrate the optimal growth problem (1) with expansive technologies in the spirit of the original von Neumann (1945) model, and its subsequent generalizations (Gale (1956), Karlin (1959)). This then brings us to the planning problem (1) above assuming that the production set is a cone, and this is precisely the model studied here together with its decentralized competitive markets representation.

It is the purpose of this paper to close the circle from early to optimal growth turnpikes, and back via unbounded growth, by offering a turnpike theorem on balanced growth paths in a model of optimal unbounded growth with many goods and a general aggregate constant returns technology. Such a theorem was anticipated by Lionel W. McKenzie in his 1998 Richard T. Ely lecture on the turnpike literature: "*To have a proof of the turnpike in a truly general convex model [of unbounded growth] it is necessary to eliminate the no joint production condition. Then the theorem would match Radner's proof of a Samuelson turnpike for the von Neumann model. Such a theorem seems to me to be within reach.*" (McKenzie (1998), p.11).[4] The comment on the no joint production condition refers to an article by Kaganovich (1998). Kaganovich shows that the optimal unbounded growth model has the turnpike property under the assumption that each firm produces only a single output (non-joint production). In this situation, it is possible to apply the method of proof of the early turnpike literature on the von Neumann-Leontief model (Morishima (1961), McKenzie (1963)). However, it is well known that the non-joint production assumption is extremely strong, excluding durable/fixed capital goods (capital goods that do not depreciate completely between periods), goods that take more than exactly one period to produce, and a typical joint output such as, one may argue, human capital.[5]

can formally be transformed to bounded growth models by rewriting to "per efficiency" or "per capita" units. Consequently such exogenous growth models are covered by the aforementioned turnpike theorems.

[4]The cited survey of McKenzie (1998) is a very good reference for understanding the context and contributions of the present paper.

[5]Of all of these, it is perhaps the exclusion of fixed capital that is the most unrealistic consequence of assuming non-joint production. Thus to quote Pierro Sraffa: "*The interest of Joint Products does not lie so much in the familiar examples of wool and mutton, or wheat and straw, as in its being the genus of which Fixed Capital is the leading species.*" (Sraffa (1960), p.63). With multiple capital goods used by a, possibly single-output firm with

Aside from the mentioned study by Kaganovich, the present paper is the first to address turnpike properties of unbounded growth models. From a technical perspective the simultaneous extension from non-joint to joint production and from stationary to non-stationary steady states is non-trivial, and requires a completely new type of turnpike proof. The gain is sizable, however, leading to the covering of a class of models for which steady states are widely focus upon in various branches of economics.

The first step in the present turnpike proof is to modify the value loss method as used by e.g. Scheinkman (1976) and McKenzie (1983) to allow for an unbounded von Neumann facet. This step may be seen as an independent contribution to the literature because it makes turnpike (and Liapunov) theory capable of handling various situations where no natural bound can be placed on zero value-loss paths. The step is indispensable in the present context because methods aimed at *detrending* an unbounded system to a bounded one (see footnote 3) generally fail unless the growth rate is exogenous. The reason for this is the following: Assume that we choose to detrend the system at some rate g. Unless we already know the average long-run growth rate (which we generally do not in endogenous growth theory), this may well either leave the economy unbounded (if g is too low), or make solutions converge toward the boundary of the commodity space (if g is too high). In the former case we have gotten nowhere, in the latter case solutions converge to zero in one or more coordinates which violates one of the most crucial assumptions in traditional turnpike proofs (cf. Yano (1984), assumption 14). Note that the latter outcome almost is unavoidable if we use the *maximum growth rate* which the system can support (and in order to avoid "insufficient detrending" we would be inclined toward doing just that).

Once the value loss method has been suitably extended, it is integrated with a recent article by Yano (1998), in which stability of the dual prices supporting the facet is proved (Yano calls this a dual stability theorem). Along the way, two theorems play a crucial role by characterizing the set of balanced growth equilibria (theorem 4), and characterizing the golden rule of unbounded growth models (theorem 5). The latter is key to understanding the present results, and it is actually somewhat surprising that existing literature has failed to address the question of what the appropriate notion of a golden rule is for unbounded growth models.

Once convergence of the dual prices to a neighborhood of a balanced price sequence has been proved, stability of the entire allocation can be proved under some extra assumptions on the mode of production. The main role of these assumptions is to rule out certain *cyclic* paths on the von Neumann facet, a necessity well known already to the earliest turnpike theorists (cf. Morishima (1961)). This second part of our results is closely related to McKenzie (1983) as well as Morishima (1961) in their by now classical papers (see also Tsukui (1967) and Yano (1998)).

The structure of the paper is as follows: Section 2 presents the model, some preliminary results, and gives a number of concrete examples of models that are covered by our results (these examples

a standard neo-classical production function, joint production is present unless all of the capital goods depreciate completely. Indeed, the firm's output will formally consist not only of its produced good but also of a depreciated quantity of each of the capital inputs. So non-joint production is violated even by the most standard models of growth when there is more than one good.

are placed in a separate subsection at the end of section 2). Section 3 contains the mentioned results on balanced growth equilibria and golden rules. Sections 4 and 5 contain statements and discussion of the stability and turnpike results. Section 6 offers concluding remarks. The main proofs are placed separately in an appendix.

2 Basic Concepts and Results

In this section we first describe the general model and define competitive equilibrium and balanced growth equilibrium (the latter being unbounded growth models' parallel to modified golden rules of bounded models). We then place assumptions on the model and establish existence of an equilibrium, a balanced growth equilibrium, and also present some results that show that allocations will always grow without bounds. Finally, a separate subsection contains a number of examples from the literature that shows the scope of this paper's results.

2.1 Basic Concepts

Time is discrete and there are $N \in \mathbf{N}$ goods at every date of the infinite time span, $t = 0, 1, 2, \ldots$. Production is described by an aggregate production set, $Y \subset \mathbb{R}_+^N \times \mathbb{R}_-^N$. A vector of inputs at date $t-1$, k_{t-1}, can be transformed into the output vector, z_t, in the following period, if and only if $(-k_{t-1}, z_t) \in Y$. A production sequence, $(-\mathbf{k}, \mathbf{z}) = (-k_{t-1}, z_t)_{t=1}^{\infty}$, is *feasible* if $(-k_{t-1}, y_t) \in Y$ at all t.

A representative consumer has preferences over consumption sequences $\bar{\mathbf{x}} = (\bar{x}_t)_{t=0}^{\infty}$, $\bar{x}_t \in \mathbb{R}_t^{N^c}$. $N^c \leq N$ is the number of consumed goods (assumed to be the first N^c goods in the indices of the production side). These preferences are assumed to be additive and stationary:

$$U(\bar{\mathbf{x}}) = \sum_{t=0}^{\infty} \delta^t u(\bar{x}_t) \tag{2}$$

Define $x_t = (\bar{x}_t, 0, \ldots, 0) \in \mathbb{R}_+^N$, $\mathbf{x} = (x_t)_{t=0}^{\infty}$, $u(x_t) \equiv u(\bar{x}_t)$, and denote the initial stock of the economy by $z_0 > 0$. Given $z_0 > 0$, an allocation $(\mathbf{x}, (-\mathbf{k}, \mathbf{z}))$ is *feasible* if $(-\mathbf{k}, \mathbf{z})$ is a feasible production sequence and $0 \leq x_t \leq z_t - k_t$ for at all t. If a feasible allocation maximizes (2) among all feasible allocations, it is *optimal.* The relevant planning problem is thus to find $(\mathbf{x}, (-\mathbf{k}, \mathbf{z}))$ that solves:

$$\begin{array}{l} \max \sum_{t=0}^{\infty} \delta^t u(x_t) \\ \text{s.t.} \begin{cases} x_t \leq z_t - k_t \\ (-k_{t-1}, z_t) \in Y, \text{all t} \\ z_0 > 0 \ \textit{given} \end{cases} \end{array} \tag{3}$$

Needless to say, (3) has no solution without further assumptions. Solutions to (3) (if any) stand in a one-to-one correspondence with competitive equilibria of a *decentralized market economy* defined next.

Let $\mathbf{p} = (p_t)_{t=0}^{\infty}$, $p_t \in \mathbb{R}_+^N$, denote a *sequence of prices*, and $p_t^c \in \mathbb{R}_+^{N^c}$ the first N^c coordinates of p_t (the price vector of the consumption goods). An aggregate production sequence, $(-\mathbf{k}, \mathbf{z})$, is *profit maximizing* if it solves:

$$\begin{array}{rl} \max & p_t z_t - p_{t-1} k_{t-1} \\ \text{s.t.} & (-k_{t-1}, z_t) \in Y \end{array} \tag{4}$$

at all dates $t = 1, 2, 3, \ldots$.[6] When Y is a cone containing the origin (assumption 2 below), this problem either does not have a solution, or aggregate profits is zero. Assuming that consumers own the initial stock it follows that the representative consumer's income W must be equal to $p_0 z_0$.

Given $\mathbf{p}$ and $W > 0$, a consumption sequence $(x_t)_{t=0}^{\infty}$ is *utility maximizing* if it maximizes (2) on the set of feasible consumption sequences, *i.e.*, the non-negative sequences for which $\sum_{t=0}^{\infty} p_t x_t \leq W$ (here it should be recalled that $p_t x_t = p_t^c \bar{x}_t$).

Definition 1 (Competitive Equilibrium) *A competitive equilibrium for the economy $\mathcal{E}(z_0) = ((u, \delta), Y, z_0)$ is a price sequence $\mathbf{p}$ and an allocation $(\mathbf{x}, (-\mathbf{k}, \mathbf{z}))$ such that (i) $\mathbf{x}$ is utility maximizing when $W = p_0 z_0$, (ii) $(-\mathbf{k}, \mathbf{z})$ is profit maximizing, and (iii) Markets clear at all dates: $x_t \leq z_t - k_t$, $t = 0, 1, 2, \ldots$.*

Under the previous assumptions, it is standard to show that if $(\mathbf{p}, (\mathbf{x}, (-\mathbf{k}, \mathbf{z}))$ is a competitive equilibrium for $\mathcal{E}(z_0)$, the allocation will solve (3) given z_0 (the first theorem of welfare economics). Conversely, if $(\mathbf{x}, (-\mathbf{k}, \mathbf{z}))$ solves (3) given z_0, $z_0 \gg 0$, there exists at least one supporting price sequence $\mathbf{p}$ such that $(\mathbf{p}, (\mathbf{x}, (-\mathbf{k}, \mathbf{z}))$ is a competitive equilibrium for $\mathcal{E}(z_0)$ (the second theorem of welfare economics in the representative consumer case).

Next, we define the critical notions of a balanced growth equilibrium (BGE) and balanced growth path (BGP):

Definition 2 (Balanced Growth) *$(\tilde{x}, (-\tilde{k}, \tilde{z}), \tilde{p}, \gamma_x, \gamma_p)$ is a balanced growth equilibrium if $\gamma_x > 1$ and the price sequence $p_t = \gamma_p^t \tilde{p}$, $t = 0, 1, 2, \ldots$ together with the allocation, $x_t = \gamma_x^t \tilde{x}$, $k_t = \gamma_x^{t+1} \tilde{k}$, $z_t = \gamma_x^t \tilde{z}$, $t = 0, 1, 2, \ldots$, is a competitive equilibrium for $\mathcal{E}(\tilde{z})$. The allocation of a balanced growth equilibrium, summarized by $(\tilde{x}, (-\tilde{k}, \tilde{z}), \gamma_x)$, is called a balanced growth path.*

Clearly, a balanced growth equilibrium (BGE) corresponds to a balanced growth path (BGP) solution of (3). The converse statement is also true: Given a BGP which solves (3), there exists a supporting balanced price sequence, $p_t = \gamma_p^t \tilde{p}$, $t = 0, 1, 2, \ldots$, hence a BGE. Note that a balanced price sequence corresponds to having a constant rate of interest given by:

$$i = \gamma_p^{-1} - 1 \tag{5}$$

We shall have a lot more to say about balanced growth and constant interest rates in later sections. Our final definition plays an important role throughout this paper.

[6]Strictly speaking this is myopic profit maximization. In a competitive equilibrium as defined in a moment, any myopic profit maximizing sequence will be profit maximizing, however.

Definition 3 (Maximum Factor of Expansion) *The maximum factor of expansion, $\lambda_* \in \mathbb{R}_+$, is defined as the largest value λ such that $z \geq \lambda k$ for some $(-k, z) \in Y$ with $z > 0$. Given λ_*, the critical discount factor is defined as the quantity λ_*^{-r}. Finally, the maximum expansion facet $\mathcal{M}_* \subseteq \mathbb{R}^N_- \times \mathbb{R}^N_+$ is the set of production plans that support maximum expansion:*

$$\mathcal{M}_* = \{(-k, z) \in Y : z \geq \lambda_* k\} \tag{6}$$

Remark 1 *Since Y is a cone (assumption 2), the condition that $z > 0$ may be replaced with the normalization that $\|z\| = 1$ in definition 3 where $\|\cdot\|$ is any norm on $\mathbb{R}^N$. From this it is easy to see that the maximum factor of expansion λ_* is a well-defined finite quantity.*[7] *It is also clear that under assumption 2, the maximum expansion facet $\mathcal{M}_*$ will be a closed, convex cone with vertex zero.*

As the name suggest, λ_* is the maximum growth factor which Y can sustain when all entries grow at the same constant rate and consumption is required to be non-negative. Indeed, for z_t and k_t to grow by some factor λ from an initial production plan $(-k, z) \in Y$ we must have: $(-\lambda^t k, \lambda^t z) \in Y$ for all t, which by constant returns is equivalent to having $(-k, z) \in Y$. Non-negatively of consumption further requires for such a production sequence that $0 \leq \lambda^t z - \lambda^{t+1} k$ for all t, which is equivalent to having $z \geq \lambda k$. The condition that $z > 0$ in definition 3 ensures that $k = z = 0$ cannot be chosen.[8] The definition of the maximum factor of expansion dates back to John von Neumann (von Neumann (1945)) who defined it in the special setting of activities models (see example 1). For the definition of the maximum expansion factor for production sets at the level of generality of this paper see Karlin (1959), Chapter 9.10.

The term *critical discount factor* is borrowed from Brock and Gale (1969) who defines this notion in exogenous growth models. When the growth rate is known (exogenously determined), the maximum factor of expansion λ_* (which is the upper bound on balanced growth) can be replaced with the exogenously given growth factor $1 + g$, and the critical discount factor taken to be $(1 + g)^{-r}$ (Brock and Gale (1969), page 230).[9] Without exogenous growth, having $\delta < 1$ will ensure that utility never becomes infinite as long as the commodity space is bounded. With positive exogenous growth, the requirement becomes instead that δ be smaller than the critical factor of discount. Thus consumers cannot discount the future any weaker than the critical factor of discount, in particular, any turnpike theorem in the exogenous growth framework (see footnote 3 in the introduction) will replace the condition that δ be sufficiently close to 1 with the condition that it be sufficiently close to the critical discount factor. The exact same thing turns out to be the case in the present setting, except that, as mentioned, we need to replace the exogenously

[7] In fact, if we take $\|z\| = \sum_n z^n$ (the one-norm), $\lambda_* = arg \max\{\lambda : z \geq \lambda k, (-k, z) \in Y, \text{ and } \|z\| = 1\}$ which is a standard convex optimization problem (this observation also makes λ_* straight-forward to compute in applications).

[8] This is of course needed to ensure that the problem makes any sense (if we allowed the choice $k = z = 0$, λ_* would not be well-defined since $(-0, \lambda 0) = (0, 0) \in Y$ for all $\lambda \geq 0$).

[9] In fact, Brock and Gale (1969) define the critical discount factor somewhat more generally by allowing also for population growth at an exogenously given rate λ. They then define the critical discount factor as $[\lambda(1 + g)]^{-r}$ where r is the elasticity of u which is of course precisely the same as u's degree of homogeneity in the present terminology.

given growth factor with the maximum factor of expansion when the critical discount factor is defined.

2.2 Assumptions and Basic Results

Consider a point $(-k, z) \in \partial Y$, where ∂Y denotes the set of points at the *boundary* of Y. Denote the *normal cone* to Y at $(-k, z)$ by $N_{(-k,z)}Y$.[10] If Y is convex, the normal cone is non-empty for all points at the boundary. Let $P(Y) = \cup_{(-k,z)\in\partial Y} N_{(-k,z)}Y$, *i.e.*, the set of vectors which generate supporting hyperplanes for Y. Finally, let $\tilde{P}(Y) = P(Y) \cap S_{+}^{2N-1}$, where $S_{+}^{2N-1} = \{(p, q) \in \mathbb{R}_{+}^{N} \times \mathbb{R}_{+}^{N} : \parallel (p, q) \parallel = 1\}$ (S_{++}^{2N-1} is defined as $S^{2N-1} \cap \mathbb{R}_{++}^{2N}$).

Let $x, y \in \mathbb{R}^N$. $x > y$ means that $x \geq y$, $x \neq y$. $x \gg y$ means that $x^n > y^n$, all $n = 1, \ldots, N$.

Assumption 1 *$u \in C^2(\mathbb{R}_{++}^{N_c}, \mathbb{R})$, is strictly concave, and $Du(x) \gg 0$, all $x \in \mathbb{R}_{++}^{N_c}$. Furthermore if $x_n = 0$, $x_m > 0$ all $m \neq n$, then $\parallel D_x u(x + \chi_n \epsilon) \parallel \to +\infty$ as $\epsilon \to 0$, $n \in \{1, \ldots, N^c\}$.*[11]

Assumption 2 *Y is a closed, convex cone with vertex zero. $(0, z) \in Y$ implies that $z = 0$, and if $(-k, z) \in Y$, then $(-k', z') \in Y$ whenever $(-k, z) \geq (-k', z')$. There exists $(-k, z) \in Y$ with $z \gg 0$. Finally, $\tilde{P}(Y)$ is a closed subset of S_{++}^{2N-1}.*

Assumption 1 is entirely standard in growth theory. It states that the instant utility function u is a twice continuously differentiable, strictly concave, and strongly monotone function that satisfies a weak Inada-type boundary condition. With the exception of the last part, assumption 2 is no less standard, saying that aggregate production possibilities exhibit constant returns, that it is impossible to produce something from nothing, that there is free disposal, and that there exists at least one production plan which yields a strictly positive output of all of the goods (the latter is just to avoid trivial cases). That $\tilde{P}(Y)$ is a closed subset of S_{++}^{2N-1} can also be thought of as an Inada-type boundary condition. In fact it will be satisfied in the standard case where Y is a differentiable manifold under monotonicity and a boundary conditions (see Yano (1998), assumption 2). Since we do not wish (or need) to impose smoothness assumptions on Y, we need a more direct assumption in order to ensure that no price can equal zero when profit is maximized, *i.e.*, we need to rule out the presence of free goods. The necessity of such an assumption is well known in turnpike theory (again see Yano (1998)). Notice that, intuitively any kind of production sector that allows firms to substitute between inputs will rule out free goods: A production plan cannot be optimal given that a price is zero when inputs can be substituted for each other because the firm would use infinitely much of the zero-cost capital input in order to produce some good with a positive price.

Since we wish to study balanced growth equilibria, we next introduce an assumption that will allow for the existence of BGEs already now (existence of an optimal program can be proved without it, but the sufficient assumptions will be more complex in its absence):

[10] The normal cone to a point $x \in X \subset \mathbb{R}^N$ is defined as $N_x X = \{q \in \mathbb{R}^N : \ q \cdot (x - y) \geq 0 \text{ for all } y \in X\}$.

[11] $\chi_n \in \mathbb{R}_{+}^{N^c}$ denotes the unit vector with 1 in the n'th coordinate and zeros everywhere else.

Assumption 3 *There exits a positive, affine transformation of* u *which is homogeneous of degree* $r < 1$.[12]

Remark 2 *The preference relation underlying the utility function* $U(\mathbf{x}) = \sum_{t=0}^{\infty} \delta^t u(x_t)$ *is invariant to positive, affine transformations of* u. *Hence it may, without loss of generality, be assumed that* u *is homogeneous of degree* $r < 1$. *With* $N^c = 1$ *this implies that* $u(x) = \frac{1}{r}x^r$ *or* $u(x) = \log x$. *If* $N^c > 1$, *the class is quite large. If for example* $N^c = 2$, *it is described by* $u(x_{1,t}, x_{2,t}) = (x_{1,t})^r f(\frac{x_{2,t}}{x_{1,t}})$, *(cf. Aczel (1966)), where* f *is an arbitrary function leading to the satisfaction of assumption 1. Assumption 3 will be further discussed at the end of this section.*

At a balanced growth equilibrium one easily shows that the rate of savings,

$$s_T = \frac{\sum_{t=T}^{\infty} p_t^c x_t - \ p_T^c x_T}{\sum_{t=T}^{\infty} p_t^c x_t} \tag{7}$$

must be constant over time. With preferences described by a discounted utility function, it is possible to show that the rate of savings will be constant whenever the rate of interest (as given by (5)) is constant *if and only if* assumption 3 is satisfied.[13] In this sense, assumption 3 is indispensable in any study of balanced growth. Still, it is reasonable to ask whether there is some independent justification for assumption 3 and so, in the end, for our focus on balanced growth in the representative consumer setting. In fact the answer is yes.

For discounted utility functions satisfying assumption 1, assumption 3 is equivalent to the requirement that the overall utility function is *homothetic*. This in turn is equivalent to assuming that consumers have linear Engel curves.[14] Thus, unless the distribution of income is restricted, there will *only* exist a representative consumer *in the first place* (Gorman (1953)) when individual consumers have identical homogeneous instant utility functions. From that perspective it may be claimed that it is the existence of a representative agent which is the truly restrictive assumption, whereas assumption 3 is just a natural consequence of this due to the simultaneously assumed additivity of preferences. On the other hand, if the distribution of income *is* restricted, say, by assuming that consumers always earn the same amount of income and have the same preferences, additivity implies the existence of a representative consumer. Insisting on that interpretation, assumption 3, hence the study of balanced growth, becomes a more questionable endeavor.[15]

The next assumption is the main condition that will ensure that the problem (3) has a solution. The critical assumption that will bound utility over the feasible set is this:

Assumption 4 $\delta < \lambda_*^{-r}$ *and* $\| z \| \leq \lambda_* \| k \|$ *all* $(-k, z) \in Y$, *where* $\| \cdot \|$ *may be any norm on* $\mathbb{R}^N$.

[12] We shall make convention that $r = 0$ if and only if u is log-linear, *i.e.*, $u(\bar{x}_t) = \sum_{n=1}^{N^c} \alpha_n \log x_t^n$.

[13] Strictly speaking, the previous statement should read "for any constant rate of interest *at the prices on the consumption goods*".

[14] These claims follow from a trivial extension of the results in Pollak (1971) and Samuelson (1961).

[15] Whether a more general concept of a balanced state can be reasonably defined and shown to posses desirable properties, is discussed briefly in section 6.

Remark 3 *Under assumption 4,* $(-k, z) \in \mathcal{M}_*$ *implies that* $z = \lambda_* k$ *(since* $\|z\| > \lambda_* \|k\|$ *would hold if* $(-k, z) \in Y$ *and* $z > \lambda_* k$*).*

The first part of assumption 4 is entirely standard and is known from exogenous growth models (see the discussion at the end of the previous subsection). The second part of assumption 4 is clearly superfluous if there is only one good. With multiple capital goods, it is needed to ensure that it is not possible to expand at a factor greater than λ_* by following some "unbalanced" production sequence. Both conditions are necessary to bound utility away from $+\infty$ over the set of feasible allocations.

Under the previous assumptions we get the following results, proofs of which can be found in Jensen (2002), chapter 4.

Theorem 1 *Under assumptions 1-4, there exists a solution to* (3)*, hence a competitive equilibrium, for all* $z_0 \gg 0$*.*

Theorem 2 *Under assumptions 1-4, there exists a* $z_0 > 0$ *such that the solution to* (3) *is a balanced growth path, hence, there exists a balanced growth equilibrium.*

Corollary 1 *In a balanced equilibrium the rate of growth lies within the bounds:* $(\lambda_* \delta)^{\frac{1}{1-r}} \leq \gamma_x \leq \lambda_*$*.*

Our final assumption ensures that optimal programs are unbounded (endogenous growth):

Assumption 5 $\lambda_* > \delta^{-1}$

Theorem 3 *Under assumptions 1-5, any solution to* (3) *will exhibit coordinatewise growth in the sense that* $\lim_{t\to\infty} x_t^n = +\infty$*, all* $n \in \{1, \ldots, N^c\}$*. In particular,* $\gamma_x > 1$ *in any balanced growth equilibrium.*

Once again, the proof is omitted to save space (and again, a detailed proof can be found in Jensen (2002)). Theorem 3 tells us that when the previous assumptions are satisfied, optimal paths can never be stationary, whether these are of the balanced growth path variety or not. This marks a clear difference from the bounded optimal growth model where the feasibility set is a subset of l_∞ (the set of supremum bounded infinite sequences).

2.3 Some Examples

Example 1 (von Neumann-Leontief technologies) *Let* $Y = \{(-k, z) : k = \mathbf{B}\eta, z = \mathbf{A}\eta, \eta \in \mathbb{R}_+^M\}$*, where* $\mathbf{A}, \mathbf{B} \in \mathbb{R}_+^{N\times M}$*,* $N \leq M$*.* $\mathbf{A}$ *is the output and* $\mathbf{B}$ *is the input matrix of the activity vector* $\eta \in \mathcal{R}_+^M$ *(von Neumann (1945)). When* $N = M = 1$*, we may without loss of generality take* $\mathbf{B} = 1$*, so this is the* $'AK'$ *model frequently studied in growth theory (cf. Rebelo (1991)). Note that for* $N > 1$ *and a discounted utility objective, the turnpike properties of such models are unexplored in the literature.*

Example 2 (Multi-sector "Non-Joint" Neo-classical Growth Models) *Assume that N sectors produce N goods by means of $M \leq N$ inputs. Let sector $n \in \{1, \ldots, N\}$ produce the n'th good according to a neo-classical production function, $z_n = f^n(k^n)$ where $k^n \in \mathbb{R}_+^M$ is the input vector. Then $Y = \{(-k, z) : z = (f^1(k^1), \ldots, f^n(k^N)), k = \sum_n k^n, \text{ and } k^n \geq 0, \text{ all } n\}$. The case $N = 2$, where the first good is human capital and the second is a combined consumption good and physical capital good has been very popular in growth theory (cf. Lucas (1990), Stokey and Rebelo (1995)). As mentioned in the introduction Kaganovich (1998) tackles this situation if one assumes that capital depreciates completely between periods (Kaganovich (1998) also assumes the each production function is strictly quasi-concave, see section 5 for further details). If not, then durable goods are present and we are in the framework of the next example.*

Example 3 (Joint Production) *Assume that a firm produces a consumption good, z_1, and that, as a by-product of this process, the firm produces a human capital good, z_2. Let physical and human capital be the inputs. This is an example of joint production. For example take, $Y = \{(-k, z) : z_1 = f^1(k_2), z_2 = f^2(k_2), k = (0, k_2) \geq 0\}$. Note that the firm cannot produce human capital without producing the physical good. This corresponds to "in-the-firm learning", while non-joint production as in the previous example corresponds better to the situation where human capital is produced in schools. Alternatively, let consumption and capital be two different goods, and assume that a firm produces consumption but that capital does not wear down completely between periods (the depreciation rate is below unity). Again, this leads to multiple outputs, i.e., joint production. Generally, such models need not converge to steady states (even though a unique locally stable steady state exists). Hence, turnpike theory is again of vital importance to establish the link between local and global stability.*

Example 4 (Homogeneous Programming) *Under the previous assumptions, the problem (3) may be cast as a homogeneous program:*

$$\begin{array}{l} \max \sum_{t=0}^{\infty} \delta^t F(z_t, z_{t+1}) \\ s.t. \begin{cases} z_{t+1} \in \Gamma(z_t) \\ z_0 > 0 \ given \end{cases} \end{array} \tag{8}$$

where $F : \mathbb{R}_+^N \times \mathbb{R}_+^N \to \mathbb{R}$ is a homogeneous, strictly concave function, and $\Gamma : \mathbb{R}_+^N \rightrightarrows \mathbb{R}_+^N$ is a convex valued correspondence and $z_{t+1} \in \Gamma(z_t)$ implies $\lambda z_{t+1} \in \Gamma(\lambda z_t)$, all $\lambda \geq 0$.

Conversely, any homogeneous program clearly corresponds to a problem of the form (3). As shown by Alvarez and Stokey (1998) solutions to homogeneous programming problems can be found by the principle of optimality and the corresponding Belmann operator. Homogeneous programs are widely applied (Alvarez and Stokey (1998) list a tremendous number of examples from the literature). For example many fertility models (cf. Barro and Becker (1989)) are of this form.

Example 5 (Unbalanced Growth) *Let $N^c = 1$, $N = 2$. Assume that one sector produces the consumption good by means of capital k_t^1 and labor, L_t, which is in fixed supply, $L_t^s = L^s$, all t. Capital is produced in a second sector by means of capital according to an $'AK'$ type technology:*

$z_{2,t+1} = Ak_t^2$. Assume that the production function of the first sector is: $z_1 = BL_t^{\alpha}(k_t^1)^{1-\alpha}$, hence homogeneous of degree 1 and separable. Inserting L^s and using market balance, $z_{1,t} = x_t$, this may be written: $x_t^{\frac{1}{1-\alpha}} = B^{\frac{1}{1-\alpha}}(L^s)^{\frac{\alpha}{1-\alpha}}k_t^1$, which is homogeneous of degree 1 in k_t^1. From the instant utility function, $u(x)$, define $\bar{u}(x) \equiv u(x^{\frac{1}{1-\alpha}})$. Let $Y = \{(-k, z) : z_1 \leq B^{\frac{1}{1-\alpha}}(L^s)^{\frac{\alpha}{1-\alpha}}k^1, z_2 \leq Ak^2, k^1 + k^2 \leq k\}$. Clearly now this problem is of the form (3), where we replace the original instant utility function with $\bar{u}$ (this will be homogeneous provided that u is homogeneous). This is a model of unbalanced growth, so called because when $x_t^{\frac{1}{1-\alpha}}$ and k_t grow with the same factor g, then x_t will expand with the factor $g^{1-\alpha}$. The previous example is taken from Rebelo (1991). Models of unbalanced growth are frequently studied in development economics, because they may be used to describe differences in growth rates among regions, patterns of migration, and other "unequal development" phenomena (see e.g. Rauch (1997)).

Example 6 (Socially Optimal Growth) *The problem (3) will describe Pareto optima of economies with externalities, imperfect competition, or other kinds of distortions. Following Boldrin, Nishimura, Shigoka and Yano (2001), let $N = 2$, $N^c = 1$, and $u(x) = x^{1-\sigma}$. The second sector produces the second good (capital) according to an $'AK'$ type technology. The first sector produces the first good (consumption) according to the production function, $z_1 = \bar{k}_2^{1-\alpha}(k_2^1)^{\alpha}$, $\alpha < 1$, where $\bar{k}_2 = k_2 = k_2^1 + k_2^2$ in equilibrium (an externality).[16] Capital depreciates at the common rate $\mu \in [0, 1]$. Let, $Y = \{(-k, z) : z_1 = k_2^{1-\alpha}(\theta k_2)^{\alpha}, z_2 = A(1-\theta)k_2 + (1-\mu)k_2, 0 \leq \theta \leq 1\}$. (3) will then describe the socially optimal allocation of the decentralized economy with externalities. As shown by Boldrin et al (2001), the decentralized economy generally displays extremely complex dynamics, including global indeterminacy and chaotic paths. Under the conditions of the theorems below, we are lead to conclude that persistent deviations from a small neighborhood of a steady state is associated with the lack of Pareto optimality, however.*

3 Balanced Growth and the "Golden Rule"

This section has two purposes. The first is to present a result that gives a complete characterization of the set of balanced growth equilibria. This result plays an important role in the proofs of all of the results that follow, and is of independent interest too. Balanced growth equilibria are unbounded growth models' direct parallel to the steady states/modified golden rules of bounded models. But what is the parallel to golden rules ? This question is the second one answered in this section. It is somewhat surprising that existing literature has left this question unanswered. For a solid turnpike theory of unbounded growth models, it is clearly inescapable because the golden rule will play the role of the turnpike (global attractor) precisely as is the case in bounded models (see McKenzie (1998) for a good discussion of this topic).

From now on we denote an economy by $\mathcal{E}^{\delta}$ rather than $\mathcal{E}(z_0)$ when we wish to make statements where the initial stock z_0 is free to vary and/or the discount factor δ plays a central role. As

[16] Boldrin et al (2001) study the case where $z_1 = \bar{k}_2^{\eta}(k_2^1)^{\alpha}$, $\eta, \alpha < 1$. Using a transformation of u as in the example of unbalanced growth above, this coincides with the case considered here.

mentioned, we begin with a result that completely characterizes the set of BGEs for a given economy $\mathcal{E}^\delta$. The (straight-forward) proof can be found in Jensen (2002), chapter 5.

Theorem 4 *$(\tilde{x}, (-\tilde{k}, \tilde{z}), \tilde{p}, \gamma_x, \gamma_p)$ is a balanced growth equilibrium of $\mathcal{E}^\delta$ if and only if it satisfies the following four conditions:*

(1) $\gamma_p = \delta\gamma_x^{r-1}$

(2) $\tilde{x} = \arg\max \{u(x) : \tilde{p}^c x \leq (1-\gamma_p\gamma_x)\tilde{p}\tilde{z} \text{ and } x \geq 0\}$

(3) $(-\tilde{k}, \tilde{z}) \in \arg\max\{\tilde{p}z - \gamma_p^{-1}\tilde{p}k : (-k, z) \in Y\}$.

(4) $\tilde{x} \leq \tilde{z} - \gamma_x\tilde{k}$

The system *(1)-(4)* is a fictitious economy which will be referred to as the *long-run economy* associated with $\mathcal{E}^\delta$. The theorem says that the set of equilibria of the long-run economy (long-run equilibria) coincides with the set of balanced growth equilibria of the original economy. In the bounded growth framework, a similar fictitious economy can be defined and shown to characterize the set of modified golden rules (Yano (1984), p. 698). Needless to say, theorem 4 provides a strong tool for the analysis of balanced growth equilibria, the long-run economy being a simple *static* system.[17] Note that by *(1)* and corollary 1, $\delta \to \lambda_*^{-r}$ implies that $\gamma_p \to \lambda_*^{-1}$ in a BGE.[18] Since the interest rate at a BGE is given by $\gamma_p^{-1} - 1$ ($\gamma_p^{-1} = 1 + i$ where i is the interest rate), this is the same as saying that the BGE interest factor converges to the maximum expansion factor λ_* as the discount factor approaches its critical upper bound. We shall be using this observation repeatedly below.

It is clear from *(1)-(4)* that a long-run equilibrium only is determined up to relative prices (so any normalization of $\tilde{p}$ is valid). Furthermore, since u is homothetic, and Y is a cone, also $(-\tilde{k}, \tilde{z})$ can be normalized arbitrarily. Intuitively, this expresses that one may "start" the economy anywhere on a balanced growth path. As mentioned several times, balanced growth equilibria are unbounded growth models' direct parallel to modified golden rules. But what is the parallel to *golden rules* ? To answer this question, let us begin by looking at bounded growth models where the golden rule is determined by solving a problem of the type,

$$\begin{aligned} &\max \; u(x) \\ &\text{s.t.} \begin{cases} x \leq z - k \\ (-k, z) \in Y \end{cases} \end{aligned} \tag{9}$$

[17] For example one can ensure uniqueness of the BGE by placing assumptions on $\mathcal{E}^\delta$ that lead to a unique long-run equilibrium. An instance of this is the case of non-joint production studied by Kaganovich (1998). Under assumption 2, it is possible to conclude from (3) that there exists exactly one level of γ_p, namely $\gamma_p = \lambda_*^{-1}$ such that this has a non-zero solution. Moreover, $\tilde{p}$ will be uniquely determined in the coordinates of produced goods (in fact, relative prices are uniquely determined whether the economy is in a BGE or not), in particular $\tilde{p}^c$ is unique (up to a normalization). By comparison with (1) the uniqueness of γ_p is seen to imply the uniqueness of the balanced growth equilibrium (up to varying prices of non-produced goods). In fact the growth factor is immediately seen to be equal to $(\lambda_*\delta)^{\frac{1}{1-r}}$, which is larger than 1 since $\lambda_* > \delta^{-1}$ by assumption 5. Finally, relative demand is independent of δ, which is seen from (2) recalling that u is homogeneous, hence homothetic. With joint production, none of the previous conclusion hold generally. For example, Jensen (2001) gives an example with $N = 2$, $N^c = 1$, where there are three BGEs, two of which are locally stable.

[18] The statement that $\delta \to \lambda_*^{-r}$ is a compact way of stating the following: For a sequence $(\delta_n)_{n=1}^\infty$ such that $\delta_n \to \lambda_*^{-r}$ as $n \to \infty$, any induced sequence of BGEs satisfies $\gamma_{x,n} \to \lambda_*$ as $n \to \infty$.

where Y is the production set *given* primary resources such as labor (which we implicitly assume are used efficiently). In the special one-sector optimal growth case where $N = 1$ and $Y = \{(-k, z) \in \mathbb{R}_- \times \mathbb{R}_+ : z \leq f(k) + (1-\delta)k\}$ ($f(k) \equiv F(k, L)$ where $L > 0$ is labor supply and $\delta \in (0, 1]$ is the rate of depreciation), this problem reduces to picking the k that maximizes $u(f(k) - \delta k)$. When u and f are differentiable, this in turn yields the familiar golden rule condition that the marginal product of capital must be equal to the rate of depreciation. The addition of exogenous technological progress and/or primary resources growth, does not change the solution in relative terms as long as u is homogeneous of some degree $r < 1$ (assumption 3) and the production set Y expands over time in a multiplicative fashion: $(-k, z) \in Y_t \Rightarrow \gamma(-k, z) \in Y_{t+1}$ where γ is the growth factor.[19]

In the endogenous growth framework where Y is a cone, it is clear that the previous optimization problem does not lead to any economically meaningful concept of a golden rule (in fact, the problem will not even have a solution). In order to arrive at a suitable golden rule notion, consider first the following problem where, intuitively, a social planner chooses the relative composition of consumption, an associated production plan, and also the welfare maximizing growth rate:

$$\begin{array}{ll} \max & \frac{1}{1-\delta\gamma_x^r} u(x) \\ \text{s.t.} & \begin{cases} x \leq z - \gamma_x k \\ (-k, z) \in Y \\ \|x\| \leq 1 \end{cases} \end{array} \tag{10}$$

Here the requirement that $\|x\| \leq 1$ is just one way to bound the problem, among many all of which are entirely equivalent (for example we might as well have taken $\|z\| = 1$). Indeed, since the idea behind (10) is to determine the "best" BGP, such normalizations merely amount to placing the economy at different points on the BGP rays at date 0. When $\delta < \lambda_*^{-r}$ (assumption 4), (10) does have a solution, so the problem is well-defined.[20] Now the golden rule of course corresponds to a solution to (10) with "no discounting" which in the present setup amounts to having δ equal to the critical discount factor λ_*^{-r} (see definition 3 and the discussion of the critical discount factor following that definition). But for that choice of δ, (10) is not well-defined.[21] The natural solution is instead to consider the limit of a sequence of problems (10) for an increasing sequence of discount factors converging to λ_*^{-r}. We first define this formally and then argue that this *does* indeed lead to the "correct" notion of a golden rule in the endogenous growth framework. It is convenient to denote the problem (10) for a given δ by $\mathcal{G}(\delta)$. Note that, reflecting the fact that an

[19] For example, in the one-sector growth model with technological progress (or labor growth), this is seen to hold when $f(k, A_t)$ is jointly homogeneous of degree 1 in (k, A_t). In this case $Y_t \equiv \{(-k, z) \in \mathbb{R}_- \times \mathbb{R}_+ : z \leq f(k, A_t) + (1-\delta)k\}$ and so $\gamma = \frac{A_{t+1}}{A_t}$ (assumed to be constant over time).

[20] Since x must be non-negative, γ_x must lie in the interval $[0, \lambda_*]$ ($(-k, \gamma_x z) \in Y$ and $\gamma_x > \lambda_*$ implies $z - k \not\geq 0$ by the definition of λ_*). So all choice variables in (10) lie in compact sets.

[21] The problems with (10) when $\delta = \lambda_*^{-r}$ are *not* limited to the fact that $\frac{1}{1-\delta\gamma_x^r}$ equals $+\infty$ when $\gamma_x = \lambda_*$ (in response, one could imagine for example solving (10) with the objective function $u(x)$, *i.e.*, removing the problematic term, but this will *not* provide a satisfactory notion of a golden rule as is explained next). Under assumption 2, $\gamma_x = \lambda_*$, $(-k, z) \in Y$, and $z \geq \lambda_* k$ will imply that $z - k = x = 0$. So any attempt at an undiscounted problem leads to an uninteresting problem where zero consumption will be chosen.

economy can be placed anywhere on a balanced growth path, we need to normalize the involved consumption and production plans in the definition.

Definition 4 (Golden Rules) *Let* $(x_j^*, (-k_j^*, z_j^*), \gamma_j^*)_{j=0}^{\infty}$ *be a sequence of solutions to* $(\mathcal{G}(\delta_j))_{j=0}^{\infty}$ *for a given increasing sequence of discount factors* $(\delta_j)_{j=0}^{\infty}$ *with limit* λ_*^{-r}*. A golden rule is defined to be any cluster point of the normalized sequence* $(\frac{1}{\|x_j^*\|}x_j^*, \frac{1}{\|(-k_j^*,z_j^*)\|}(-k_j^*, z_j^*), \gamma_j^*)_{j=0}^{\infty}$.

Lemma 1 *Let* $(x^*, (-k^*, z^*), \gamma^*)$ *be a golden rule. Then* $\gamma^* = \lambda_*$ *(the maximum factor of expansion),* x^* *is uniquely determined (it depends only on* Y *and* u*), and* $(-k^*, z^*)$ *is an element of the maximum expansion facet* $\mathcal{M}_*$.

Proof. It is obvious that $\gamma^* = \lambda_*$ (once way to see this is to observe that as δ^j converges to λ_*^{-r}, the partial derivative of the objective function taken with respect to γ_x converges to $+\infty$). Since clearly $\frac{1}{\|(-k_j^*,z_j^*)\|}(-k_j^*, z_j^*) \in Y$ and $z_j^* \geq \gamma_j^* k_j^*$ for all j, it immediately follows that $\lim_{j\to\infty} \frac{1}{\|(-k_j^*,z_j^*)\|}(-k_j^*,\, z_j^*) \in \mathcal{M}_+$. It remains to be shown that $\frac{1}{\|x_j^*\|}x_j^*$ has a unique limit point that is independent of the specific sequences chosen. Let $\gamma_j^* \to \lambda_*$ be the sequence of optimally chosen growth factors and let $\frac{1}{\|x_j^*\|}x_j^*$, $j = 0, 1, 2, \ldots$, be the associated solutions to the problems (10) where γ_j^* are taken as given (so think of this as the second stage of a two-stage problem where we first choose the growth factors and then the consumption and production plans). Since for γ_j^* given, the problem (10) consists of maximizing a strictly concave function on a convex set, it has a unique solution for each j. By the theorem of the maximum, this solution depends continuously on γ_j^* as well as δ_j, in particular $\frac{1}{\|x_j^*\|}x_j^*$ converges to a (uniquely determined) limit point as γ_j^* converges to λ_* and δ_j converges to λ_*^{-r}. It is clear that this limit point does not depend on the original sequences since no matter which sequence we begin with, we always have $\gamma_j^* \to \lambda_*$ and $\delta_j \to \lambda_*^{-r}$, and these limit points are all that matter for the determination of $\lim_{j\to\infty} \frac{1}{\|x_j^*\|}x_j^*$. ■

The following theorem shows that just as modified golden rules converge to golden rules in bounded growth models, balanced growth paths converge to golden rules in the endogenous growth setting (in either case when the discount factor converges to its critical upper bound which is 1 in the bounded setting and λ_*^{-r} in the present setting).

Theorem 5 *Let* $(\tilde{x}^j,\, (-\tilde{k}^j,\, \tilde{z}^j), \gamma_x^j)_{j=1}^{\infty}$ *be a sequence of BGPs for a given increasing sequence of discount factors* $(\delta_j)_{j=0}^{\infty}$ *with limit* λ_*^{-r}*. Then the BGP sequence converges to a golden rule in the sense that* $\lim_{l\to\infty} \gamma_x^{j_l} = \lambda_*$, $\lim_{l\to\infty} \frac{1}{\|\tilde{x}^{j_l}\|}\tilde{x}^{j_l} = x^*$*, and* $\lim_{l\to\infty} \frac{1}{\|(-\tilde{k}^{j_l},\tilde{z}^{j_l})\|}(-\tilde{k}^{j_l},\, \tilde{z}^{j_l}) \in \mathcal{M}_*$ *for any subsequence* $(\tilde{x}^{j_l},\, (-\tilde{k}^{j_l},\, \tilde{z}^{j_l}), \gamma_x^{j_l})_{l=1}^{\infty}$ *such that the limits are well-defined.*

Proof. To simplify notation we are going to index the subsequence of the theorem by j rather than j_l. We begin with the claim that $\frac{1}{\|\tilde{x}^j\|}\tilde{x}^j \to x^*$ where x^* is the uniquely determined golden rule consumption vector of lemma 1. Let $\hat{x}^j$ denote the (unique) solution to (10) given δ_j. To arrive at a contradiction, imagine that $\frac{1}{\|\tilde{x}^j\|}\tilde{x}^j$ does not converge to x^* as $j \to \infty$. Since each $\hat{x}^j$ uniquely solves (10) and $\frac{1}{\|\hat{x}^j\|}\hat{x}^j \to x^*$, it follows that $u(\hat{x}^j) \geq u(\frac{1}{\|\tilde{x}^j\|}\tilde{x}^j) + \epsilon$, all n where $\epsilon > 0$.

This in turn implies that $\tilde{p}^j \hat{x}^j \geq \tilde{p}^j \tilde{x}^j + \rho$, all j where $\rho > 0$. Since $\hat{x}^j = \hat{z}^j - \gamma_x^j \hat{k}^j$, it follows that for all j, there exists $(\hat{k}^j, \hat{z}^j) \in Y$ such that $\tilde{p}^j[\hat{z}^n - \gamma_x^j \hat{k}^j] \geq \rho > 0$. But this leads to the following contradiction: $\tilde{p}^j(\gamma_p^j z - k) \leq 0$ for all j and all $(k, z) \in Y$ by (3). Since $\gamma_x^j \to \lambda_*$, it follows from *(1)* that $\gamma_p^j \to \lambda_*^{-1}$. Hence $\lim_{j\to\infty} \tilde{p}^j(\gamma_p^j \hat{z}^j - \hat{k}^j) = \lambda_*^{-1} \lim_{j\to\infty} \tilde{p}^j(\hat{z}^j - \lambda_* \hat{k}^j) \geq \lambda_*^{-1}\rho > 0$ (here we are possibly passing to yet another subsequence to ensure convergence, if so we index it again by j). But for sufficiently large j, the latter implies the existence of some $(-k, z) \in Y$ and $\tilde{p}^j(\gamma_p^j z - k) > 0$. A contradiction. ■

Theorem 5 says that as consumers become increasingly patient, relative consumption at a BGP converges to the unique golden rule consumption vector x^* of lemma 1. Since by the Euler conditions $\frac{1}{\|\tilde{p}^{c,j}\|}\tilde{p}^{c,j} = \frac{1}{\|Du(\tilde{x}^j)\|} Du(\tilde{x}^j)$ (for all j of a sequence of BGEs), it follows immediately from theorem 5 that as $\delta \to \lambda_*^{-r}$ the consumption prices at a BGE converge to a (after a normalization) uniquely determined *golden rule consumption price vector*:

$$p^{c,*} \equiv \frac{1}{\|Du(x^*)\|} Du(x^*) \tag{11}$$

To repeat, x^* and $p^{c,*}$ are uniquely determined and depend on Y and u alone (in particular, they are independent of the specific sequence of discount factors considered). x^* will - precisely as is the case with golden rules of bounded growth models - have the interpretation of solving the "best possible consumption problem" in the limit where the discount factor approaches its critical upper bound. $p^{*,c}$ is the uniquely determined vector of support prices (precisely, it is uniquely determined because the utility function is assumed to be differentiable). Unfortunately, one cannot in the same way determine uniquely either k, z, or the vector of non-consumed goods' prices in the limit. What can be said is that $(-k, z) \in \mathcal{M}_*$ (the maximum expansion facet), and furthermore, considering (3) of theorem 4 as $\gamma_p \to \lambda_*^{-1}$ (as it will when $\delta \to \lambda_*^{-r}$), it is seen that $\tilde{p}$ must as $\delta \to \lambda_*^{-r}$ converge to a p^* such that:

$$(-k^*, z^*) \in arg\max\{p^* z - \lambda_* p^* k : (-k, z) \in Y\} \tag{12}$$

Rewriting slightly, this gives rise to the following definition:

Definition 5 *A price vector $p^* \in \mathbb{R}_+^N$ is called a von Neumann price vector if:*

$$p^* z - \lambda_* p^* k \leq 0 \text{ for all } (-k, z) \in Y. \tag{13}$$

Note that if p^* is a von Neumann price vector then $(p_{t-1}, p_t) = (\lambda_* p^*, p^*)$ supports any production plan on the maximum expansion facet as a profit maximizing production sequence (*i.e.*, (13) will hold with equality when $(-k, z) \in \mathcal{M}_*$ is chosen).[22] As the name suggests, the von Neumann price vector is precisely the vector of support prices in the special case of the von Neumann model, and more generally it is the vector of support prices with associated interest

[22]Here is why: When $(-\tilde{k}, \tilde{z}) \in Y$ and $\tilde{z} \geq \lambda_* \tilde{k}$, it must hold that $p^* \tilde{z} - \lambda_* \tilde{z} \geq p^* \lambda_* \tilde{k} - \lambda_* p^* \tilde{k} = 0$. Combine with (13) to conclude that $(-\tilde{k}, \tilde{z})$ yields maximum profit (namely 0).

factor $1+i = \lambda_* \; (= \gamma_p^{-1})$ in general models of balanced growth without consumption in the spirit of von Neumann (1945) (see Karlin (1959), Chapter 9, especially Theorem 9.10.2).

In summary, the BGE price variables $(\tilde{p}, \gamma_p)$ will as $\delta \to \lambda_*^{-r}$ go toward (p^*, λ_*^{-1}) where p^* is a von Neumann price vector. And just as production plans are in general *not* uniquely determined at the golden rule, the von Neumann price vector will in general not be uniquely determined. Note that *if* p^* is unique (which as seen from definition 5 depends solely on the structure of Y), a kind of "non-substitution theorem" governs BGP consumption when δ is close to λ_*^{-r} since $p^{c,*}$ uniquely determines x^* via equation (11). Thus, when p^* (and so $p^{c,*}$) is uniquely determined from the production side, preferences play no role in the determination of relative consumption at and near the golden rule. Such uniqueness of p^* holds under non-joint production, but we stress that in general it does not.[23]

4 The Price and Consumption Turnpikes

In this section we present our main global stability and turnpike results. These are all valid at the very high level of generality of assumptions 1-5, in particular, no special/non-standard assumptions are placed on the aggregate production set Y. The "cost" is that we shall *not* be able to predict global convergence of production sequences but "only" of prices and consumption sequences. The turnpike in any of our turnpike results will be found to be the golden rule(s) of the previous section further underscoring the relevance and correctness of our golden rule definition. In the next section we turn to the more delicate question of the capital turnpike which forces us to place stronger assumptions on the production set. All results in the present section are new, and as mentioned in the introduction, they are based on the approach of Yano (1998) altered to accommodate unbounded growth.

Since turnpike theorems require that consumers discount the future "sufficiently weakly", it will come as no surprise that the critical condition for our results is that δ is "sufficiently" close to the critical discount factor λ_*^{-r} of definition 3.[24] In fact, as mentioned at the end of section 2.2, the requirement that δ be close to the critical discount factor is also the key condition in turnpike theorems for exogenous growth models.

Given $\delta \in \Delta$, an associated BGE for $\mathcal{E}^\delta$, and a competitive equilibrium price sequence for some $z_0 \gg 0$, $\mathbf{p} = (p_t)_{t=0}^\infty$, define the *detrended competitive equilibrium price sequence*: $\hat{p}_t = \gamma_p^{-t} p_t$, $t = 0, 1, 2, \ldots$. The next theorem states that this price sequence will, eventually, be in an arbitrarily small neighborhood of the relative price vector $\tilde{p}$ of a BGE, provided that the future is sufficiently weakly discounted. Since the term "turnpike theorem" is normally reserved for results that predict convergence to the unique limit of a golden rule (see *e.g.* Yano (1998) for a discussion of this topic), we follow Yano (1998)'s use of terminology and refer to this result as a *dual stability*

[23]For a proof that $\mathcal{M}_*$ as well as p^* are unique up to a normalization under non-joint production see Kaganovich (1998), Lemma 6).

[24]If $r > 0$, λ_*^{-r} will be strictly below unity so in this case unbounded growth turnpikes actually requires "less patience" than turnpikes in bounded models. If on the other hand $r < 0$, "upcounting" ($\delta > 1$) may be necessary. Note that such "upcounting" is not in disagreement with the previous assumptions when $r < 0$.

theorem ("dual" refers to the fact that it is a statement about prices rather than allocations).[25]

Theorem 6 (Dual Stability Theorem) *Let assumptions 1-5 be satisfied. Then for any* $\alpha > 0$ *there exists a level of discount* $\delta_\alpha < \lambda_*^{-r}$*, and a date* $T \in \mathbf{N}_0$ *such that whenever* $\delta \geq \delta_\alpha$ *the following stability result holds: For arbitrary initial stock* $z_0 > 0$*, an optimal solution to* (3) *given* z_0*,* $(\mathbf{x}, (-\mathbf{k}, \mathbf{z}))$*, is supported as a competitive equilibrium by a price sequence,* $\mathbf{p} = (p_t)_{t=0}^{\infty}$ *such that:*

$$\| b\hat{p}_t - \tilde{p} \| \leq \alpha \tag{14}$$

all $t \geq T$*, where* $\hat{p}_t = \gamma_p^{-t} p_t$*,* $t = 0, 1, 2, \ldots$*,* γ_p *and* $\tilde{p}$ *are those of a balanced growth equilibrium of* $\mathcal{E}^\delta$*,* $\| \tilde{p} \| = 1$*, and* b *is a positive constant.*

Note that γ_p and $\tilde{p}$ correspond to *some* BGE. Hence, if there are multiple BGPs, their supporting prices sequences must be "close" to each other by the previous theorem and the triangle inequality. Since the support prices of an optimal program are generally not unique in the last $N - N^c$ coordinates (p_t^c, $t = 0, 1, 2, \ldots$, is uniquely determined due to assumption 1), this means that the price sequence with the property of the theorem much be suitably chosen. Next we have:

Theorem 7 (Consumption Turnpike) *For* $\alpha > 0$*, take* δ_α *and* T *as in theorem 6. Then, if* $\delta \geq \delta_\alpha$*:*

$$\| c\hat{x}_t - x^* \| \leq \alpha \tag{15}$$

for all $t \geq T$*, where* $\hat{x}_t = \gamma_x^{-t} x_t$*,* $t = 0, 1, 2, \ldots$*;* γ_x *is that of a balanced growth equilibrium for* $\mathcal{E}^\delta$*,* x^* *is the unique (normalized) golden rule consumption vector of lemma 1, and* c *is a positive constant.*

Proof. From the Euler conditions:

$$\eta p_t^c = Du(x_t) \tag{16}$$

where $\eta > 0$ is the marginal utility of consumption (Lagrange multiplier). So p_t^c uniquely determines x_t at each date. Since by theorem 6, $\hat{p}_t^c \to \tilde{p}^c$ (the BGE consumption price vector) as $\delta \to \lambda_*^{-r}$, it follows that $a\hat{x}_t \to \tilde{x}$ where a is a positive constant. But by theorem 5, $\tilde{x} \to x^*$ as $\delta \to \lambda_*^{-r}$. The conclusion of the theorem now follows from the triangle inequality. ■

This theorem has as an immediate corollary the following dual turnpike theorem:

Corollary 2 (Dual Consumption Turnpike) *The statement of theorem 6 remains valid if* (14) *is replaced with:*

$$\| b\hat{p}_t^c - p^{c,*} \| \leq \alpha \tag{17}$$

for all $t \geq T$*, where* $p^{c,*}$ *is the unique (normalized) golden rule consumption price vector as given by* (11)*.*

[25] The proof is placed in the appendix. It should be noted that all arguments which replicate arguments from existing turnpike proofs have been left out to keep the proof at its absolute minimum length.

Note that while theorem 6 is a "stability theorem" as discussed immediately prior to its statement, the previous two results are "true" turnpike theorems predicting convergence to uniquely determined golden rule vectors (here uniquely determined means that they depend only on Y and u).

What about the remaining $N - N^c$ prices of the non-consumed goods (if there are any non-consumed goods!) ? By theorem 6, prices will eventually get arbitrarily close to $\tilde{p}$ of a BGE provided that the future is sufficiently weakly discounted. As shown at the end of the previous section, $\tilde{p}$ will in turn approach a von Neumann price vector (definition 5) as $\delta \to \lambda_*^{-r}$. Hence we have:

Theorem 8 (Dual Turnpike) *The statement of theorem 6 remains valid if* (14) *is replaced with:*

$$\| b\hat{p}_t - p^* \| \leq \alpha \tag{18}$$

for all $t \geq T$, where p^* is a von Neumann price vector whose first N^c coordinates are equal to $p^{c,*}$.

It should be stressed that the von Neumann price vector p^* of theorem 8 will in general *not* be uniquely determined in the $N - N^c$ coordinates of the non-consumed goods. So what the previous theorem says is that as consumers become sufficiently patient, any BGP can be supported by *any* price sequence whose relative price vector is close to *any* von Neumann price vector. Evidently, if $\mathcal{M}_*$ is such that the supporting von Neumann price vector is unique up to a normalization *given* that the first N^c coordinates must equal the uniquely determined golden rule consumption price vector $p^{c,*}$, then this price vector will play the role of a dual turnpike price vector effectively pinning down the prices of the non-consumed goods.[26]

5 The Capital Turnpike

In the previous section it was established that prices and consumption paths converge to those of a BGE if future consumption is discounted sufficiently weakly by consumers. Balanced consumption and price sequences may, however, coexist with fluctuating capital sequences. In order to rule this out, hence prove a *capital turnpike* result, more structure must be placed on the aggregate production set. Before we get to such assumptions, let us state a general result that parallels McKenzie (1983) who proves convergence of production sequences to a neighborhood of the von Neumann facet in bounded growth models. First, we need to define an appropriate notion of the von Neumann facet for the unbounded growth framework.

Definition 6 (The von Neumann Facet) *Let $\tilde{x}$, γ_x, $\tilde{p}$, and γ_p be those of a BGE. The von Neumann facet, $\mathcal{N}_* \subseteq \mathbb{R}^N_- \times \mathcal{R}^N_+$, is defined as those production plans that satisfy (3) and (4) of*

[26] Assuming that the von Neumann price vector is unique is precisely what McKenzie (1983) does when he proves a dual turnpike theorem in the bounded growth framework. Even though there are, of course, substantial differences between that framework and the present one with unbounded growth, the discussion relating to uniqueness of prices at the "golden rule" is essentially the same in the two.

theorem 4, i.e., those production plans $(-\hat{k},\hat{z}) \in Y$ such that: (i) $(-\hat{k},\hat{z}) \in arg\max\{\tilde{p}z - \gamma_p^{-1}\tilde{p}k : (-k,z) \in Y\}$, and (ii) $\tilde{x} = \hat{z} - \gamma_x\hat{k}$.

Theorem 9 (Stability of the von Neumann Facet) *Let assumptions 1-5 be satisfied. Then for any $\alpha > 0$ there exists a level of discount $\delta_\alpha < \lambda_*^{-r}$, and a date $T \in \mathbf{N}_0$ such that whenever $\delta \geq \delta_\alpha$ the following stability result holds: For arbitrary initial stock $z_0 > 0$, the detrended capital sequence of an optimal solution $(\mathbf{x},(-\mathbf{k},\mathbf{z}))$ to (3) will eventually be in an α-neighborhood of the von Neumann facet of any BGE, i.e.,*

$$\inf_{(-k,z)\in\mathcal{N}_*} \| (-\hat{k}_t,\hat{z}_t) - (-k,z) \| \leq \alpha \tag{19}$$

all $t \geq T$, where $(-\hat{k}_t,\hat{z}_t) = (-\gamma_x^{-(t+1)}k_t, \gamma_x^{-t}z_t)$ all t, γ_x is the growth factor of some BGE, and $\mathcal{N}_*$ is the von Neumann facet associated with that same BGE.

Proof. The result is a direct consequence of theorems 4, 6 and 7. ■

It is easy to verify that as $\delta \to \lambda_*^{-r}$ the von Neumann facet converges (in the set theoretic sense) to the maximum expansion facet $\mathcal{M}_*$, or more accurately we should say that the von Neumann facet of *any* BGE converges to $\mathcal{M}_*$ (there may be multiple BGEs). It follows then that if the maximum expansion facet is *trivial*, *i.e.*, if $\mathcal{M}_*$ consists of a single ray, we get a turnpike theorem predicting that the detrended production sequence converges to the detrended capital sequence of a BGE. To save space, we do not state this result as a theorem even though it is of course an important one. There are at least two important cases where the maximum expansion facet is trivial: One is under non-joint production with strictly quasi-concave production functions (see Kaganovich (1998), Lemma 6), the other is when the aggregate production set is a *strictly convex cone* (Radner (1961)) as defined in a moment. So in each of these cases, turnpike theorems are established. Obviously, it is also perfectly possible that production sequences that lay on the von Neumann facet must converge to the BGP production plan and if so we once again get a global stability theorem. The following result can actually be interpreted in this way, *i.e.*, as a way to ensure that even when the von Neumann facet is not trivial, capital sequences will converge to those of a BGP. Evidently, to ensure this outcome we must place more structure on the aggregate production set Y, and unfortunately, the conditions ensuring a turnpike theorem become much less “aesthetic” to look at. Since we set out from the beginning to cover such models as the original von Neumann model with consumption and a discounted utility objective (where, to be sure, the maximum expansion facet need not be trivial), dealing with this issue seems however, to be inescapable.

An aggregate production set, Y, is said to *disaggregate* into M sectors if it is possible to break Y into $M \in \mathbf{N}$ independent sectors, each of which is represented by a production set $T^m \subset \mathbb{R}_-^N \times \mathbb{R}_+^N$, $m = 1,\ldots,M$. Y then, is the direct sum $Y = \sum_{m=1}^M T^m$, *i.e.*, $(-k,z) \in Y$ if and only if there exist $((-k^1,z^1),\ldots,(-k^M,z^M))$ such that $k = \sum_m k^m$, $z = \sum_m z^m$, and $(-k^m,z^m) \in T^m$, all m. Of course, it may happen that the only possible disaggregation is the trivial one, $M = 1$ and

$T^1 = Y$, which intuitively means that there is only one firm in the economy. We hurry on to an assumption, which must be satisfied for *some* disaggregation:

Assumption 6 *For each sector $m \in \{1, \ldots, M\}$ there is a good, $n_m \in \{1, \ldots N\}$, which is produced only by the m'th sector in equilibrium. Moreover, T^m is a strictly convex cone for all m, i.e., if $(-k, z)$ and $(-\tilde{k}, \tilde{z})$ are both in T^m, the (feasible) production plan $(-k^\alpha, z^\alpha) = \alpha(-k, z) + (1-\alpha)(-\tilde{k}, \tilde{z})$, $0 < \alpha < 1$, is in the interior of T^m, or else $(-k^m, z^m) = \beta(-\tilde{k}^m, \tilde{z}^m)$, for some $\beta \geq 0$.*

The trivial disaggregation $M = 1$ and $T^1 = Y$ always satisfies the first part of assumption 6 (existence of an output that is produced by only that sector). If the trivial disaggregation also satisfies the last part of assumption 6, Y is a strictly convex cone and in this case we already know that a turnpike theorem holds. Assumption 6 implies, among other things, that $M \leq N$ for some disaggregation, and that $M = N$ if and only if every sector produces exactly one good (non-joint production). Under non-joint production, the production function of sector m must, moreover, be *strictly quasi-concave* for T^m to be a strictly convex cone (cf. Kaganovich (1998), assumption 4). Again, this is not an interesting special case as far as the following results are concerned because a turnpike theorem covering it was already established above.

Choose a price pair, (p_t, p_{t+1}), such that (4) has a non-trivial solution. Under assumption 6, (p_t, p_{t+1}) determines an input vector, $a^m(p_t, p_{t+1}) = (a_1^m, \ldots, a_N^m)(p_t, p_{t+1}) \in \mathbb{R}_+^N$, and an output vector, $b^m(p_t, p_{t+1}) = (b_1^m, \ldots, b_N^m)\ (p_t, p_{t+1}) \in \mathbb{R}_+^N$, such that sector m maximizes profits if and only if it chooses $(-k^m, z^m) \in \{(-k^m, z^m) \ : \ k^m = \lambda^m a^m(p_t, p_{t+1}),\ z^m = \lambda^m b^m(p_t, p_{t+1}),\ \lambda^m \in \mathbb{R}_+\} \subset T^m$. Define matrices $\mathbf{A}(p_t, p_{t+1}) = [a^1(p_t, p_{t+1}), \ldots, a^M(p_t, p_{t+1})] \in \mathbb{R}_+^{N \times M}$, and $\mathbf{B}(p_t, p_{t+1}) = [b^1(p_t, p_{t+1}), \ldots,\ b^M(p_t, p_{t+1})] \in \mathbb{R}_+^{N \times M}$, and let $\lambda = (\lambda^1, \ldots, \lambda^M) \in \mathbb{R}_+^M$. It follows that for $(p_t, p_{t+1}) \in P$ (the normal surface), $(-k^*, z^*) \in Y$ is a solution to (4) if and only if $(-k^*, z^*) \in \{(-k, z) \ : \ k = \mathbf{A}(p_t, p_{t+1})\lambda, z = \mathbf{B}(p_t, p_{t+1})\lambda\ , \lambda \in \mathbb{R}_+^M\}$.

A particularly simple case is von-Neumann-Leontief technologies, where $Y = \{(-k, z) : \ k = \mathbf{A}\lambda, z = \mathbf{B}\lambda\ , \lambda \in \mathbb{R}_+^M\}$ for *fixed* matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}_+^{N \times M}$. In this particular case, the assumptions below trace directly to $\mathbf{A}$ and $\mathbf{B}$ and consequently they are easy to verify. But in the general case where sectors have more than a single technique to choose from, the matrices $\mathbf{A}(p_t, p_{t+1})$ and $\mathbf{B}(p_t, p_{t+1})$ will vary with (p_t, p_{t+1}).

Note that, due to the first part of assumption 6, $\mathbf{B}(p_t, p_{t+1})$ is *triangular* (after suitably rearranging the rows). Hence, we may pick a submatrix $\tilde{\mathbf{B}}(p_t, p_{t+1}) \in \mathbb{R}^{M \times M}$ which has *full rank*. Let $\tilde{\mathbf{A}}(p_t, p_{t+1}) \in \mathbb{R}^{M \times M}$ denote the submatrix of $\mathbf{A}(p_t, p_{t+1})$ consisting of the same rows as $\tilde{\mathbf{B}}(p_t, p_{t+1})$. For a balanced growth equilibrium, $(\tilde{x}, (-\tilde{k}, \tilde{z}), \tilde{p}, \gamma_x, \gamma_p)$, consider the matrix $\mathbf{Q}^{\gamma_x}(\tilde{p}, \gamma_p \tilde{p}) \equiv \gamma_x \tilde{\mathbf{B}}(\tilde{p}, \gamma_p \tilde{p})^{-1} \tilde{\mathbf{A}}(\tilde{p}, \gamma_p \tilde{p})$. If this matrix has full rank M, and none of its M eigenvalues have modulus equal to 1, we shall say that the BGE is *regular*. We return to this below, but first let us state our second main result:

Theorem 10 *Let all previous assumptions be satisfied and assume that the economy has a regular balanced growth equilibrium. Then, for any $\alpha > 0$ there exist $\delta_\alpha < \lambda_*^{-r}$ and $T \in \mathbf{N}_0$ such that if*

$\delta \geq \delta_\alpha$, then, regardless of $z_0 > 0$, an optimal production path, $(-\mathbf{k}, \mathbf{z})$, will satisfy:

$$\parallel \frac{1}{b_t}(z_t, k_t) - (\tilde{z}, \tilde{k}) \parallel < \alpha \text{ and } \frac{b_{t+1}}{b_t} \in (\gamma_x - \alpha, \gamma_x + \alpha) \tag{20}$$

for all $t \geq T$, where γ_x, $\tilde{z}$, and $\tilde{k}$, are those of a balanced growth equilibrium for $\mathcal{E}^\delta$ with $\parallel \tilde{k} \parallel = 1$.

Take $\alpha > 0$, and $\delta \geq \delta_\alpha$ as in the theorem, but assume that all BGEs of $\mathcal{E}^\delta$ are irregular (*i.e.*, not regular). In this situation theorem 10 looses its validity. Indeed, if $\mathbf{Q} = \mathbf{Q}^{\gamma_x}(\tilde{p}, \gamma_p \tilde{p})$ has eigenvalues on the unit sphere it is straight-forward to construct examples where the supporting price sequence actually *converges* to the BGE price sequence, but where the production sequence is characterized by undampened cyclic motions which leave the α-neighborhood periodically. Comparing with e.g. Tsukui (1967) it will be seen that the same phenomenon can arise even without discounting, hence regularity of the BGEs is definitely an indispensable assumption. The case where $\mathbf{Q}$ has rank smaller than M is more special, and may be compatible with convergence into a neighborhood of the BGE (this statement should be read as a conjecture). A complete study of this situation would be interesting, but hardly interesting enough to be included on the present pages. It should also be noted that various assumptions (e.g. local uniqueness of long-run equilibria) imply regularity of BGEs, in whole or in part. That each sector produces a good which no other sector produces (assumption 6), is actually in this spirit, implying that $\mathbf{B}$ has rank M. If $\mathbf{B}$ does not have full rank, the situation is similar to the situation where $\mathbf{A}$, hence $\mathbf{Q}$, has reduced rank.

6 Concluding Remarks

In the present paper we have proved a number of global stability and turnpike theorems for a general model of optimal unbounded growth. The model is very general at the production side allowing for joint production and non-smooth technologies such as von Neumann-Leontief activity models. The results establish a direct link between early turnpike theorems (e.g. Morishima (1961) and Tsukui (1967)), and turnpike theorems on optimal growth models (e.g. Scheinkman (1976)). Moreover, they provide the first global stability results on "new growth models" that are general enough to include the various models studied within this field (see section 2.3).

The turnpike results of this paper implies that the widespread focus on steady states in the new growth literature is not *ad hoc*. More practically framed, the implication is that when modelers choose a very simple economic structure which is globally stable, it is not the implicitly assumed stability properties which should raise concern.[27] Alternatively one can view the present results as a *theoretical* explanation of why reality appears to confine to (neighborhoods of) steady states. Of course there are many more reasons why turnpike results have received the attention of so many scholars (McKenzie (1968) contains a longer discussion of those). To mention just one, any planner - be it an investment planner, a growth theorist, or some other person who wishes to study

[27] Of course this statement is only true given the assumptions of this paper.

properties of optimal paths - is greatly benefited by the fact that *any* optimal program, regardless of initial conditions, converge to the neighborhood of paths (steady states) which are very simple to compute. If this were not the case, the planner would have to have excessive knowledge of the distant future (or the growth theorist would have to assume it on part of the agents). If the model has the turnpike property, this information requirement is relaxed to knowledge about tastes and technology until a not so distant future when the system has approached a steady state. Among other things this makes optimal programs *implementable* and theoretical models *realistic*. From a computational perspective, it implies that a sufficiently long *finite* problem with *arbitrary* initial conditions, will approximate the actual solution. This, of course, is the same as saying that little information is required on part of the agents who are assumed to actually do the calculations.

While the turnpike theorem is surely the main result of this paper, the other results are also important, and together they constitute a fairly complete characterization of the unbounded optimal growth model.[28] Yet, there are at least three generalizations which would seem worth while to make. The first is to leave the representative consumer framework of the optimal growth model in favor of a multiple consumer general equilibrium model. This issue is dealt with in Jensen (2006) (the global stability theorem in that paper builds directly on theorem 6 of the present paper). The second and much more challenging generalization is to dismiss assumption 3, which states that the instant utility function is *homogeneous* (see the discussion at the end of section 2). Without this assumption, BGEs will generally not exist, and another candidate class for the attracting allocations must be found or another type of turnpike result must be aimed at (see McKenzie (1968) for such alternatives). If this is possible, an integration with e.g. Gantz (1980) might lead to a list of very interesting turnpike results. In particular one might wish for some sort of 'approximate' balancedness of the turnpike under suitable assumptions. Finally, the environment with no uncertainty is obviously restrictive; but again a generalization would in all likelihood be far from easy.

7 Appendix

7.1 Proof of theorem 6

Pick an arbitrary initial stock, $z_0 \gg 0$, and keep this fixed throughout the proof. Let $\Delta \subset (0,1)$ denote the set consisting of those discount factors for which assumptions 4 and 5 are satisfied given λ_*. In the following δ is varied within this set, and when a competitive equilibrium (CE) or a balanced growth equilibrium (BGE) is mentioned, it is to be understood that this is for the economy $\mathcal{E}^\delta(z_0)$ as parameterized by $\delta \in \Delta$ (when confusion can arise the dependence on δ will be made explicit).

By theorem 2 there exists a BGE, $(\tilde{x}, (-\tilde{k}, \tilde{z}), \tilde{p}, \gamma_x, \gamma_p)$. Theorem 4 describes the set of such BGEs (given $\delta \in \Delta$). Pick some BGE, $(\tilde{x}, (-\tilde{k}, \tilde{z}), \tilde{p}, \gamma_x, \gamma_p)$. We then say that $p \in \mathbb{R}^N_+$, $\| p \| = 1$, *supports the BGE production plan* $(-\tilde{k}, \tilde{z})$, if (3) of theorem 4 (long-run profit maximization), is

[28] The results of Alvarez and Stokey (1998) may be claimed to complete the characterization.

satisfied for p. The set of such price vectors is denoted $\tilde{P}_{(-\tilde{k},\tilde{z})}$. Clearly $\tilde{p} \in \tilde{P}_{(-\tilde{k},\tilde{z})}$, but in the absence of differentiability assumptions on Y, $\tilde{P}_{(-\tilde{k},\tilde{z})}$ need not be a singleton. In the argument below, it is understood that a BGE price vector has been picked from $\tilde{P}_{(-\tilde{k},\tilde{z})}$ ((N.2) below normalizes the price vector to unity, which creates correspondence with the previous construction).

By theorem 1 there exists a CE, $(\mathbf{p}, (\mathbf{x}, (-\mathbf{k}, \mathbf{z}))$. Define the *detrended* CE, $\hat{\pi}$, by: $\hat{x}_t = \gamma_x^{-t} x_t$, $\hat{k}_t = \gamma_x^{-(t+1)} k_t$, $\hat{z}_t = \gamma_x^{-t} z_t$, and $\hat{p}_t = (\delta \gamma_x^{r-1})^{-t} p_t$, $t = 0, 1, 2, \ldots$.

As mentioned after theorem 4, the BGE is subject to two normalizations; of $\tilde{p}$ and $(-\tilde{k}, \tilde{z})$, respectively. We shall take:

$$(N.1) \quad \| \tilde{k} \| = 1$$

$$(N.2) \quad \| \tilde{p} \| = 1$$

It is straight forward to show that the detrended BGE will satisfy the following 'detrended Euler condition' condition:

$$\tilde{\eta} \tilde{p}^c = Du(\tilde{x}) \tag{21}$$

where $\tilde{\eta} > 0$ is the Lagrange-multiplier.

The detrended CE price sequence $(\hat{p}_t)_{t=0}^{\infty}$ is also subject to a normalization. Thus, we may scale this sequence such that:

$$\tilde{\eta} \hat{p}_t^c = Du(\hat{x}_t), \; \textit{all } t \tag{22}$$

where $\tilde{\eta}$ is the Lagrange multiplier from (21).

Define the *implicit factor of discount* by:

$$\hat{\rho} = \delta \gamma_x^r \tag{23}$$

Let $\tilde{\Delta} = (\delta \gamma_x^r \mid \delta \in \Delta$ and γ_x is the rate of growth induced by $\delta\}$. Clearly $\tilde{\Delta}$ is the set of implicit factors of discount which are possible to obtain by varying $\delta \in \Delta$. Note by (1) of theorem 4:

$$\hat{\rho} = \delta \gamma_x^{r-1} \gamma_x = \gamma_p \gamma_x \tag{24}$$

As an immediate consequence of this and corollary 1 we record (see also the discussion at the beginning of section 2):

Sublemma 11 *Let $(\delta_m)_{m=0}^{\infty}$ be any sequence for which $\delta_m \to \lambda_*^{-r}$ as $m \to \infty$. Let $(\hat{\rho}_m)_{m=0}^{\infty}$ denote a sequence of associated implicit factors of discount (i.e., where $\hat{\rho}_m = \delta_m \gamma_{x,m}^r$, and $\gamma_{x,m}$ is that of the long-run equilibrium for $\mathcal{E}^{\delta_m}$). Then $\hat{\rho}_m < 1$ for all m, and $\hat{\rho}_m \to 1$ as $m \to \infty$.*

For $a > 0$ define:

$$L_t^a = -(a^{r-1} \hat{p}_t - \tilde{p})(a \hat{k}_t - \tilde{k}) \tag{25}$$

and from this the *Liapounov function*

$$L_t = \min_{\alpha \geq 0} \; L_t^{\alpha \|\hat{p}_t\|^{\frac{1}{1-r}}} \tag{26}$$

When this is well-defined, define α_t implicitly by: $L_t = L_t^{\alpha_t \|\hat{p}_t\|^{\frac{1}{1-r}}}$ (the α that is a minimizer in (26)).

$$G_t^a = [(\tilde{\eta}^{-1}u(a\hat{x}_t) - \tilde{p}a\hat{x}_t) - (\tilde{\eta}^{-1}u(\tilde{x}) - \tilde{p}\tilde{x}) + a(\tilde{p}\hat{z}_t - \hat{\rho}^{-1}\tilde{p}\hat{k}_{t-1})] \tag{27}$$

and,

$$\hat{G}_t^a = [(\tilde{\eta}^{-1}u(a\hat{x}_t) - a^r\hat{p}_t\hat{x}_t) - (\tilde{\eta}^{-1}u(\tilde{x}) - a^{r-1}\hat{p}_t\tilde{x}) - a^{r-1}(\hat{p}_t\tilde{z} - \hat{\rho}^{-1}\hat{p}_{t-1}\tilde{k})] \tag{28}$$

Clearly, L_t^a, G_t^a, and $\hat{G}_t^a$ are well defined for all t and $\hat{\rho}$. By Jensen (2002), appendix C, lemma 7: $G_t^a \leq 0$ all t, $a > 0$. By a similar argument, it may be shown that $\hat{G}_t^a \geq 0$.

Sublemma 12

$$\hat{\rho}^{-1}L_t^a - L_{t+1}^a = \hat{G}_{t+1}^a - G_{t+1}^a \tag{29}$$

for all $a > 0$, $t \in \mathbf{N}_0$, and $\delta \in \Delta$. Moreover,

$$L_t^a \geq 0 \tag{30}$$

for all a and t. Finally, L_t is well-defined and non-negative.

Proof: In competitive equilibrium at date t, $p_t z_t - p_{t-1}k_{t-1}$ will be maximized over all $(-k_{t-1}, z_t) \in Y$. Using the detrending conditions above, this objective may be rewritten as $\hat{\rho}^{-t}\hat{p}_t\hat{z}_t - \hat{\rho}^{-(t-1)}\hat{p}_{t-1}\hat{k}_{t-1}$. This is clearly equivalent to the maximization of $\hat{p}_t\hat{z}_t - \hat{\rho}^{-1}\hat{p}_{t-1}\hat{k}_{t-1}$, which thus equals zero (Y is a cone, and after detrending: $(-\hat{k}_{t-1}, \gamma_x\hat{z}_t) \in Y$ becomes the constraint). Similarly $\tilde{p}\tilde{z} - \hat{\rho}^{-1}\tilde{p}\tilde{k} = 0$ in the BGE (compare with (3) of theorem 4). Use these zero-profit conditions together with market balance $\hat{x}_t = \hat{z}_t - \hat{k}_t$, and, $\tilde{x} = \tilde{z} - \tilde{k}$, to see that:

$$G_t^a = [u(a\hat{x}_t) - u(\tilde{x})] + a(\tilde{p}\hat{k}_t - \hat{\rho}^{-1}\tilde{p}\hat{k}_{t-1}) - (\tilde{p}\tilde{k} - \hat{\rho}^{-1}\tilde{p}\tilde{k})$$

$$\hat{G}_t^a = [u(a\hat{x}_t) - u(\tilde{x})] + a^r(\hat{p}_t\hat{k}_t - \hat{\rho}^{-1}\hat{p}_{t-1}\hat{k}_{t-1}) - a^{r-1}(\hat{p}_t\tilde{k} - \hat{\rho}^{-1}\hat{p}_{t-1}\tilde{k})$$

Subtract to get:

$$[\hat{G}_{t+\tau}^a - G_{t+\tau}^a] = (a^{r-1}\hat{p}_{t+\tau} - \tilde{p})(a\hat{k}_{t+\tau} - \tilde{k}) - \hat{\rho}^{-1}(a^{r-1}\hat{p}_{t+\tau-1} - \tilde{p})(a\hat{k}_{t+\tau-1} - \tilde{k})$$

Set $\tau = 1$ and (29) follows. Now sum over the previous equation:

$$\sum_{\tau=1}^{T} \hat{\rho}^\tau[\hat{G}_{t+\tau}^a - G_{t+\tau}^a] = -(a^{r-1}\hat{p}_t - \tilde{p})(a\hat{k}_t - \tilde{k}) + \hat{\rho}^T(a^{r-1}\hat{p}_{T-1} - \tilde{p})(a\hat{k}_{T-1} - \tilde{k})$$

Clearly: $\hat{\rho}^T(a^{r-1}\hat{p}_{T-1} - \tilde{p})(a\hat{k}_{T-1} - \tilde{k}) \leq \hat{\rho}^T(a^r\hat{p}_{T-1}\hat{k}_{T-1} + \tilde{p}\tilde{k})$. 'Retrend' and use that $\hat{\rho}^t = (\gamma_p\gamma_x)^t$ to see that this is equal to: $\hat{\rho}^{-1}a^r p_{T-1}z_{T-1} + \hat{\rho}^T\tilde{p}\tilde{k}$. As for the last term this clearly goes to zero as $T \to \infty$ ($\hat{\rho} < 1$). It may also be shown that the first term goes to zero as $T \to \infty$. Indeed $\lim_{T\to\infty} p_T z_T \to 0$ is the *transversality condition* of the planning problem, which is indeed satisfied under the present assumptions. In conclusion, $\lim_{T\to\infty} \hat{\rho}^T(a^{r-1}\hat{p}_{T-1} - \tilde{p})(a\hat{k}_{T-1} - \tilde{k}) \in \mathbb{R}_- \cup \{-\infty\}$. Since $\hat{G}_{t+\tau}^a - G_{t+\tau}^a \geq 0$, $\sum_{\tau=1}^{T} \hat{\rho}^\tau[\hat{G}_{t+\tau}^a - G_{t+\tau}^a]$ is non-decreasing in T, hence

$\hat{\rho}^T(a^{r-1}\hat{p}_{T-1}-\tilde{p})(a\hat{k}_{T-1}-\tilde{k})$ is non-decreasing in T. A non-decreasing sequence with an upper bound converges, hence $\lim_{T\to\infty}\hat{\rho}^T(a^{r-1}\hat{p}_{T-1}-\tilde{p})(a\hat{k}_{T-1}-\tilde{k})=\epsilon$, $\epsilon\le 0$. This implies that also $\sum_{\tau=1}^{\infty}\hat{\rho}^\tau[\hat{G}^a_{t+\tau}-G^a_{t+\tau}]$ exists. Thus, $-(a^{r-1}\hat{p}_t-\tilde{p})(a\hat{k}_t-\tilde{k})=\sum_{\tau=1}^{\infty}\hat{\rho}^\tau[\hat{G}^a_{t+\tau}-G^a_{t+\tau}]-\epsilon\ge 0$. But the left-hand side is L^a_t, and so, $L^a_t\ge 0$, all a,t, so the second claim of the lemma has been proved. By definition, $L_t=\min_{a\ge 0}L^a_t$. It is easily shown that $L^a_t\to+\infty$ if either $a\to 0$ or $a\to+\infty$. But then, since $L^a_t\ge 0$ for all a, the minimizer may be sought within a compact interval, and existence follows by an application of Weierstrass' theorem. It follows that L_t is well-defined. That $L_t\ge 0$ is a trivial consequence of the non-negativity of L^a_t. **Q.E.D.**

Sublemma 13 *There exists $\bar{L}>0$ such that $L_t\le\bar{L}$ for all $t\in\mathbf{N}_0$ and $\delta\in\Delta$.*

Proof: By definition of L_t: $L_t\le L^a_t$, for all $a\ge 0$. In particular,

$$L_t\le-(\alpha^{r-1}\frac{\hat{p}_t}{\parallel\hat{p}_t\parallel}-\tilde{p})(\alpha(\frac{1}{\parallel\hat{p}_t\parallel})^{\frac{1}{r-1}}\hat{k}_t-\tilde{k}) \tag{31}$$

for all $\alpha\ge 0$. By the last part of assumption 2 it is possible to show that there exists, α, independent of t and δ, such that: $\alpha^{r-1}\frac{\hat{p}_t}{\|\hat{p}_t\|}-\tilde{p}\ge 0$, all t, δ. Indeed, by $(N.2)$, $\tilde{p}$ is uniformly coordinatewise bounded from above, and $\tilde{P}(Y)$ is a closed, hence compact, subset of S^{2N-1}_{++}, bounding $\frac{\hat{p}_t}{\|\hat{p}_t\|}$ uniformly coordinatewise from below. But then, since by the previous lemma, $L^a_t\ge 0$, for all a and t:

$$L^{\bar{\alpha}a_t}_t\le(\alpha^{r-1}_t\frac{\hat{p}_t}{\parallel\hat{p}_t\parallel}-\tilde{p})\tilde{k} \tag{32}$$

By $(N.1)$, $\tilde{k}$ is clearly uniformly bounded from above (and below), and the lemma follows since, by the previous observations, *all* entries at the right-hand side are then uniformly bounded. **Q.E.D.**

Since L_t is well-defined for all t, a sequence of minimizers, $(\alpha_t)_{t=0}^{\infty}$, such that $L_t=L_t^{\alpha_t\|\hat{p}_t\|^{\frac{1}{1-r}}}$, $t=0,1,2,\ldots$, exists (in fact this sequence is unique, because L^a_t is a strictly concave function in a). Consequently we may define:

$$\delta^{\hat{\rho}}_t=\hat{G}_t^{\alpha_{t-1}\|\hat{p}_{t-1}\|^{\frac{1}{1-r}}}-G_t^{\alpha_{t-1}\|\hat{p}_{t-1}\|^{\frac{1}{1-r}}} \tag{33}$$

For all $t=0,1,2,\ldots$. By definition of L^a_t it is straight forward to show that there exists $\bar{\alpha}>0$ such that $L_t^{\alpha\|\hat{p}_t\|^{\frac{1}{1-r}}}\le(\alpha^{r-1}\frac{\hat{p}_t}{\|\hat{p}_t\|}-\tilde{p})\tilde{k}$ for all $\alpha\le\bar{\alpha}$ (see the proof of lemma 13 for a similar argument). But then, for the upper bound of lemma 13 not to be violated, there must exist $\underline{\alpha}>0$ such that $\alpha_t\ge\underline{\alpha}$ for all t and δ. That α_t is also bounded from above follows from essentially the same argument. In conclusion $(\alpha_t)_{t=0}^{\infty}$ is a sequence for which $\alpha_t\in[x,y]$, $x>0$, $y<+\infty$, irrespectively of t and δ. We are now ready to prove a crucial lemma:

Sublemma 14 (Value Loss lemma)

For all $\alpha>0$, there exist $\beta>0$ and $b>0$ such that for all $\delta\in\Delta$ and $t\in\mathbf{N}_0$, $\delta^{\hat{\rho}}_t\le\beta$ implies:

$$\parallel b(\hat{p}_t,\hat{\rho}^{-1}\hat{p}_{t-1})-(\tilde{p},\hat{\rho}^{-1}\tilde{p})\parallel\le\alpha \tag{34}$$

where $\| \tilde{p} \| = 1$, *and* $\tilde{p}$ *supports a BGE production plan associated with* δ *in the sense of* (3) *of theorem 4.*

Proof: From the definition of $\delta_t^{\hat{\rho}}$, G_t^a, and $\hat{G}_t^a$ (see the proof of lemma 12):

$$\delta_t^{\hat{\rho}} \geq -[\alpha_{t-1}^{r-1}(\frac{\hat{p}_t}{\| \hat{p}_{t-1} \|}\tilde{z} - \hat{\rho}^{-1}\frac{\hat{p}_{t-1}}{\| \hat{p}_{t-1} \|}\tilde{k})] \tag{35}$$

Assume that the conclusion of the lemma is false. Then there exist $\alpha > 0$, and a sequence $(t_m, \hat{\rho}_m)_{m=0}^{\infty}$, such that $\delta_{t_m}^{\hat{\rho}_m} \to 0$, as $m \to \infty$, while $\| b(\hat{p}_{t_m}, \hat{\rho}_m^{-1}\hat{p}_{t_m-1}) - (\tilde{p}_m, \hat{\rho}_m^{-1}\tilde{p}_m) \| > \alpha$, for all $b > 0$, and all m. Considering the last expression of (35), it is clear, however, that all entries have convergent subsequences from the sequence induced by $m = 0, 1, 2, \ldots$, *i.e.*, there exist $\breve{a} > 0$, $\breve{p}_t \gg 0$, $\breve{p}_{t-1} \gg 0$, $\breve{p} \gg 0$, $\breve{z} > 0$, $\breve{k} > 0$, and $\breve{\rho} \leq 1$, such that: $\alpha_{t-1} \to \breve{a}$, $\frac{\hat{p}_{t_m}}{\|\hat{p}_{t_m-1}\|} \to \breve{p}_t$, $\frac{\hat{p}_{t_m-1}}{\|\hat{p}_{t_m-1}\|} \to \breve{p}_{t-1}$, $\tilde{p}_m \to \breve{p}$, $\tilde{z}_m \to \breve{z}$, $\tilde{k}_m \to \breve{k}$, and $\hat{\rho}_m \to \breve{\rho}$, as $m \to \infty$ (w.l.g. index the subsequence by m again). That all price limits are strictly positive follow directly from the last part of assumption 2. That $\breve{z} > 0$ is a consequence of $(N.1)$. Since Y is closed, $(-\breve{k}, \breve{\gamma}_x\breve{z}) \in Y$, where $\breve{\gamma}_x$ denotes the limit of $(\gamma_{x,m})_{m=0}^{\infty}$ (note that ρ_m is induced by some δ_m and the growth rate of a BGE as determined from theorem 4 by: $\hat{\rho}_m = \delta_m\gamma_{x,m}^r$. The limit of $\gamma_{x,m}$ exists by corollary 1. But $(\breve{k}, \breve{\gamma}_x\breve{z}) \in Y$, $\breve{z} > 0$ implies $\breve{k} > 0$ (assumption 2). Since $\breve{a} > 0$, it holds for this limit that:

$$\breve{p}_t\breve{z} - \breve{\rho}^{-1}\breve{p}_{t-1}\breve{k} = 0 \tag{36}$$

Since Y is closed and $\breve{p} \gg 0$, it can also be shown that:

$$0 = \breve{p}\breve{z} - \breve{\rho}^{-1}\breve{p}\breve{k} \geq \breve{p}z - \breve{\rho}^{-1}\breve{p}k \tag{37}$$

for all $(-k, z) \in \{(-k, z) \in \mathbb{R}_-^N \times \mathbb{R}_+^N : (-k, \breve{\gamma}_x z) \in Y\}$. But since $\| b(\breve{p}_t, \breve{\rho}^{-1}\breve{p}_{t-1}) - (\breve{p}, \breve{\rho}^{-1}\breve{p}) \| > \alpha$, all $b > 0$, and for all price pairs $(\breve{p}, \breve{\rho}^{-1}\breve{p})$ which support $(\breve{z}, \breve{k})$ in the sense of (37), we have arrived at a contradiction, because (36) states that $(\breve{p}_t, \breve{\rho}^{-1}\breve{p}_{t-1})$ lies in this support set. **Q.E.D.**

We now need two lemmas on the connection between the value loss function, $\delta_t^{\hat{\rho}}$, and the Liapounov function, L_t.

Sublemma 15 *For all* $\beta > 0$ *there is* $\alpha > 0$ *such that* $\delta_t^{\hat{\rho}} < \alpha \Rightarrow L_t < \beta$, *for all* $t \in \mathbf{N}_0$ *and all* $\delta \in \Delta$.

Proof: Assume that this is not the case. Then there exists a sequence $(\delta_m, t_m)_{m=1}^{\infty}$ such that $L_t \geq \beta$ for all m while $\delta_{t_m}^{\hat{\rho}_m} \to 0$ as $m \to \infty$. But by lemma 14 this implies that $\| b_m\hat{p}_{t_m} - \tilde{p}_m \| \to 0$, for some sequence $(b_m, \tilde{p}_m)_{m=1}^{\infty}$, $b_m > 0$ all m, where $\tilde{p}_m$ supports the BGE production plan associated with δ_m. Since by $(N.2)$, $\| \tilde{p}_m \| = 1$, this implies that $\frac{1}{\|\hat{p}_{t_m}\|}\hat{p}_{t_m} \to \breve{p}$, for a convergent subsequence (indexed by m again), where $\breve{p}$ denotes the limit point of $\tilde{p}_m$. As explained prior to the proof of lemma 14, α_t lays in a compact interval for all δ and t, hence $\alpha_{t_m} \to \alpha > 0$ for yet another subsequence. In fact, by the previous argument, it is clear that $\alpha = 1$, hence $\alpha_{t_m}^{r-1}\frac{1}{\|\hat{p}_t\|}\hat{p}_t \to \breve{p}$. But then, $L_{t_m} \to 0$ as $m \to \infty$. This is a contradiction, having assumed that $L_t \geq \beta > 0$ for all m. **Q.E.D.**

Sublemma 16 *For all $\alpha > 0$: $L_t < \hat{\rho}\alpha \Rightarrow \delta_{t+1}^{\hat{\rho}} < \alpha$, for all $t \in \mathbf{N}_0$ and all $\hat{\rho} \in \tilde{\Delta}$.*

Proof: By lemma 12 and the non-negativity of the Liapounov function: $\hat{\rho}^{-1} L_t^a \geq \hat{G}_{t+1}^a - G_{t+1}^a$ for all $a > 0$. Take $a = \alpha_t \parallel \hat{p}_t \parallel^{\frac{1}{1-r}}$, and the conclusion of the lemma follows immediately from the definitions. **Q.E.D.**

Next, it is shown that, for $\hat{\rho}$ sufficiently close to 1, the detrended CE price will eventually 'visit' a neighborhood of the BGE price.

Sublemma 17 (Visit Lemma)

For all $\alpha > 0$ there is $0 < \hat{\rho}_\alpha < 1$ and $T \in \mathbf{N}_0$ such that

$$\delta_t^{\hat{\rho}} \leq \alpha$$

for at least one $t \leq T$ provided that $\hat{\rho} \geq \hat{\rho}_\alpha$.

Proof: Proof by contradiction. Assume that for some $\beta > 0$, $\delta_t^{\hat{\rho}} > \beta$ for all t and $\hat{\rho}$. For convenience set $a_t = \alpha_t \parallel \hat{p}_t \parallel^{\frac{1}{1-r}}$. By lemma 12: $L_t^{a_t} - \hat{\rho} L_{t+1}^{a_t} > \hat{\rho}\beta$, all t. Clearly, $L_{t+1}^{a_t} \geq L_{t+1}$ (the latter is the minimum), and so, since $L_t^{a_t} = L_t$: $L_t - \hat{\rho} L_{t+1} > \hat{\rho}\beta$, all t. Rewrite this as $L_t - L_{t+1} > (\hat{\rho} - 1) L_{t+1} + \hat{\rho}\beta \geq (\hat{\rho} - 1)\bar{L} + \hat{\rho}\beta$ where $\bar{L}$ is the (uniform) upper bound on L_t of lemma 13. Now sum from $t = 0$ to T: $L_0 - L_{T+1} > T[(\hat{\rho} - 1)\bar{L} + \hat{\rho}\beta]$. But the left-hand side is smaller than $\bar{L}$ so we get: $\bar{L} > T[(\hat{\rho} - 1)\bar{L} + \hat{\rho}\beta]$. Clearly now if $\hat{\rho}$ is such that $(\hat{\rho} - 1)\bar{L} + \hat{\rho}\beta > 0$ there exists a $T > 0$ such that this inequality is violated. We conclude as in the lemma. **Q.E.D.**

Using lemma 17 together with lemmas 15 and 16, it is now possible to "trap" $\delta_t^{\hat{\rho}}$ such that $\delta_t^{\hat{\rho}} \leq \alpha$ for all $t \geq T$, provided that $\hat{\rho}$ is sufficiently large (see e.g. McKenzie (1983) for details). By lemmas 11 and 14 this implies the following:

Sublemma 18 *For any $\alpha > 0$ there exist $\delta_\alpha < \lambda_*^{-r}$, $b > 0$, and $T \in \mathbf{N}_0$ such that $\delta \geq \delta_\alpha$ implies that for all $t \geq T$:*

$$\parallel b(\hat{p}_t, \hat{\rho}^{-1}\hat{p}_{t-1}) - (\tilde{p}, \hat{\rho}^{-1}\tilde{p}) \parallel \leq \alpha \tag{38}$$

The formulation of theorem 6 is an immidiate consequence of this lemma.

7.2 Proof of theorem 10

Due to space constraints, the following proof is kept at its absolute minimum length. As in the foregoing proof, let $\Delta \subset (0, 1)$ denote the set consisting of those discount factors for which assumptions 4 and 5 are satisfied given λ_*. Take arbitrary $z_0 \gg 0$ and consider a solution to the planning problem (3) given $\delta \in \Delta$, $(\mathbf{x}, (-\mathbf{k}, \mathbf{z}))$. Pick a supporting price sequence $(p_t)_{t=0}^{\infty}$. Under assumption 6, there will exist a sequence $(\lambda_t)_{t=1}^{\infty}$, $\lambda_t \in \mathbb{R}_+^M$, all t, such that:

$$z_t = \mathbf{B}(p_{t-1}, p_t)\lambda_t \tag{39}$$

and,

$$k_{t-1} = \mathbf{A}(p_{t-1}, p_t)\lambda_t \tag{40}$$

at all $t = 0, 1, 2, \ldots$. To simplify notation, let $\mathbf{A}_t \equiv \mathbf{A}(p_{t-1}, p_t)$, and $\mathbf{B}_t = \mathbf{B}(p_{t-1}, p_t)$. Market clearing at date t may now be written:

$$\mathbf{B}_t\lambda_t = x_t + \mathbf{A}_{t+1}\lambda_{t+1} \tag{41}$$

where we set $\mathbf{B}_0\lambda_0 = z_0$. Define $\hat{\lambda}_t = \frac{1}{\|x_t\|}\lambda_t$, $\hat{x}_t = \frac{x_t}{\|x_t\|}$, and rewrite this as:

$$\mathbf{B}_t\hat{\lambda}_t = \hat{x}_t + \frac{\| x_{t+1} \|}{\| x_t \|}\mathbf{A}_{t+1}\hat{\lambda}_{t+1} \tag{42}$$

Now, consider a balanced growth equilibrium $(\tilde{x}, (-\tilde{k}, \tilde{z}), \tilde{p}, \gamma_x, \gamma_p)$, associated with the same $\delta \in \Delta$. Let $\mathbf{A}^* = \mathbf{A}(\tilde{p}, \gamma_p\tilde{p})$ and $\mathbf{B}^* = \mathbf{B}(\tilde{p}, \gamma_p\tilde{p})$. Consider the system:

$$\mathbf{B}^*\tilde{\lambda}_t = \tilde{x} + \gamma_x\mathbf{A}^*\tilde{\lambda}_{t+1} \tag{43}$$

Define the norm of a matrix $\mathbf{M} \in \mathbb{R}^{N\times M}$ by:

$$\| \mathbf{M} \| = \max_{n=1,\ldots,N,\ m=1,\ldots,M} | \mathbf{M}_{n,m} |$$

It is easily shown that $\mathbf{A}(p_t, p_{t+1})$ and $\mathbf{B}(p_t, p_{t+1})$ are homogeneous of degree 0, and continuous on $\tilde{P}(Y)$, hence from theorem 6 (see lemma 18), and corollary 7 we get:

Sublemma 19 *For any $\alpha > 0$ there exists $\delta_\alpha < \lambda_*^{-r}$ and $T \in \mathbf{N}$, such that $\delta \geq \delta_\alpha$ implies that:*

$$\| \mathbf{A}_t - \mathbf{A}^* \| \leq \alpha \tag{44}$$

$$\| \mathbf{B}_t - \mathbf{B}^* \| \leq \alpha \tag{45}$$

$$\| \hat{x}_t - \tilde{x} \| \leq \alpha \tag{46}$$

and,

$$\left| \frac{\| x_{t+1} \|}{\| x_t \|} - \gamma_x \right| \leq \alpha \tag{47}$$

for all $t \geq T$, where $\| \tilde{x} \|$ is chosen such that (46) is satisfied (i.e., $c = 1$ in corollary 7).

Since the balanced growth path is supported by the balanced price sequence, (43) has a stationary solution, *i.e.*, there exists $\tilde{\lambda} \in \mathbb{R}_+^M$ such that:

$$\mathbf{B}^*\tilde{\lambda} = \tilde{x} + \gamma_x\mathbf{A}^*\tilde{\lambda} \tag{48}$$

Since $\mathbf{B}^*$ is triangular by assumption 6, there exists a submatrix consisting of M linearly independent rows from $\mathbf{B}^*$. Call this $\tilde{\mathbf{B}}^* \in \mathbb{R}_+^{M\times M}$. Choose the same rows from $\mathbf{A}^*$, and denote the submatrix by $\tilde{\mathbf{A}}^* \in \mathbb{R}_+^{M\times M}$. (43) then implies:

$$\tilde{\lambda}_t = (\tilde{\mathbf{B}}^*)^{-1}\tilde{x} + \gamma_x\mathbf{Q}\tilde{\lambda}_{t+1} \tag{49}$$

where $\mathbf{Q} = (\tilde{\mathbf{B}}^*)^{-1}\tilde{\mathbf{A}}^*$. By assumption the BGE is *regular*, hence $\mathbf{Q}$ has full rank (equivalently, $\tilde{\mathbf{A}}^*$, is invertible), and $\gamma_x\mathbf{Q}$ has no eigenvalues with modulus 1. This allows us to conclude that $\tilde{\lambda}$ is a global saddle point of the system (43) (cf. Tsukui (1967)). Moreover, given $\tilde{\lambda}_0$, a solution to (43), $(\tilde{\lambda}_t)_{t=1}^{\infty}$, will either converge to $\tilde{\lambda}$, or move away from $\tilde{\lambda}$ (eventually) along the unstable manifold of the saddle. Picking α of lemma 19 sufficiently small, and $\delta \geq \delta_\alpha$, it is then easily shown (again, see Tsukui (1967) for a similar argument), that for $t \geq T$, $\| b\hat{\lambda}_t - \tilde{\lambda}_t \| \leq \gamma\alpha$, where b is a positive constant, and γ is a positive constant which is independent of α and T.

The rest of the proof is now trivial: $\tilde{\lambda}_t$ must converge to $\tilde{\lambda}$ for if it did not, it would "drive" $\hat{\lambda}_t$ with it along the unstable manifold of the saddle, eventually violate feasibility. But then $\hat{\lambda}_t$ will eventually be in a neighborhood of $\tilde{\lambda}$, hence the detrended production plan will be in a neighborhood of $(-\tilde{k}, \tilde{z})$. This finishes the proof.

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