G53: Supplemental notes on Meucci exercises

Colin Rowat
Room 220, J.G. Smith Building
c.rowat@bham.ac.uk
www.socscistaff.bham.ac.uk/rowat

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These notes supplement the other texts and course material: they are not intended as a complete guide. For a full treatment, consult the references listed in the lecture notes.

The references below are to the 15 August 2010 version of Meucci's exercises.

2.2.4

6.2.1

The Meucci (2010f) reference doesn't seem to help. The definitions of the generalised r-square in The Lab (and the attachment to my 2 May 2017 e-mail to the msc-mat-fin list) refer to $\boldsymbol{\sigma}^2$ instead of $\boldsymbol{W}$. As the solution presented in the exercises does not contain $\boldsymbol{W}$ (or $\boldsymbol{\sigma}^2$), I ignore it here.

The problem is to

$$\max_{\boldsymbol{B}} R^2 \{\boldsymbol{Y}, \boldsymbol{X}\} \equiv 1 - \frac{tr\left(Cov\left\{\boldsymbol{Y} - \boldsymbol{X}\right\}\right)}{tr\left(Cov\left(\boldsymbol{X}\right)\right)};$$

which — as it only contains one term in $\boldsymbol{B}$ — reduces to $\min_{\boldsymbol{B}} tr\left(Cov\left\{\boldsymbol{BF}-\boldsymbol{X}\right\}\right)$. By (M2.67) and some tedious algebra, this expands to:

$$
\begin{aligned}
&\min_{\boldsymbol{B}}\ tr\left(E\left\{(\boldsymbol{BF}-\boldsymbol{X}-E\{\boldsymbol{BF}-\boldsymbol{X}\})(\boldsymbol{BF}-\boldsymbol{X}-E\{\boldsymbol{BF}-\boldsymbol{X}\})'\right\}\right)\\
=&\min_{\boldsymbol{B}}\ tr\left(E\left\{(\boldsymbol{BF}-\boldsymbol{X}-E\{\boldsymbol{BF}-\boldsymbol{X}\})(\boldsymbol{F}'\boldsymbol{B}'-\boldsymbol{X}'-E\{\boldsymbol{F}'\boldsymbol{B}'-\boldsymbol{X}'\})\right\}\right)\\
=&\min_{\boldsymbol{B}}\ tr\left(E\left\{(\boldsymbol{BF}-\boldsymbol{X}-E\{\boldsymbol{BF}-\boldsymbol{X}\})(\boldsymbol{F}'\boldsymbol{B}'-\boldsymbol{X}'-E\{\boldsymbol{F}'\boldsymbol{B}'-\boldsymbol{X}'\})\right\}\right)\\
=&\min_{\boldsymbol{B}}\ tr\left(E\left\{\boldsymbol{BFF}'\boldsymbol{B}'-\boldsymbol{BFX}'-\boldsymbol{BF}E\{\boldsymbol{F}'\boldsymbol{B}'-\boldsymbol{X}'\}-\boldsymbol{XF}'\boldsymbol{B}'+\boldsymbol{XX}'+\boldsymbol{X}E\{\boldsymbol{F}'\boldsymbol{B}'-\boldsymbol{X}'\}-E\{\boldsymbol{BF}-\boldsymbol{X}\}\boldsymbol{F}'\boldsymbol{B}'+E\{\boldsymbol{BF}-\boldsymbol{X}\}\boldsymbol{X}'+E\{\boldsymbol{BF}-\boldsymbol{X}\}E\{\boldsymbol{F}'\boldsymbol{B}'-\boldsymbol{X}'\}\right\}\right)\\
=&\min_{\boldsymbol{B}}\ tr\left(\boldsymbol{B}E\{\boldsymbol{FF}'\}\boldsymbol{B}'\right)-tr\left(\boldsymbol{B}E\{\boldsymbol{FX}'\}\right)-tr\left(\boldsymbol{B}E\{\boldsymbol{F}\}E\{\boldsymbol{F}'\}\boldsymbol{B}'\right)+tr\left(\boldsymbol{B}E\{\boldsymbol{F}\}E\{\boldsymbol{X}'\}\right)-tr\left(E\{\boldsymbol{XF}'\}\boldsymbol{B}'\right)+tr\left(E\{\boldsymbol{XX}'\}\right)+tr\left(E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}\boldsymbol{B}'\right)-tr\left(E\{\boldsymbol{X}\}E\{\boldsymbol{X}'\}\right)\\
&-\ tr\left(\boldsymbol{B}E\{\boldsymbol{F}\}E\{\boldsymbol{F}\}'\boldsymbol{B}'\right)+tr\left(E\{\boldsymbol{X}\}E\{\boldsymbol{F}\}'\boldsymbol{B}'\right)+tr\left(\boldsymbol{B}E\{\boldsymbol{F}\}E\{\boldsymbol{X}\}'\right)-tr\left(E\{\boldsymbol{X}\}E\{\boldsymbol{X}\}'\right)+tr\left(\boldsymbol{B}E\{\boldsymbol{F}\}E\{\boldsymbol{F}'\}\boldsymbol{B}'\right)-tr\left(\boldsymbol{B}E\{\boldsymbol{F}\}E\{\boldsymbol{X}'\}\right)\\
&-\ tr\left(E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}\boldsymbol{B}'\right)+tr\left(E\{\boldsymbol{X}\}E\{\boldsymbol{X}'\}\right)
\end{aligned}
$$

Rules for differentiating with traces (the Frobenius norm) can be found in §2.5 of The Matrix Cookbook. For conformable matrices $\boldsymbol{A},\boldsymbol{C},\boldsymbol{D}$ and $\boldsymbol{G}$, the relevant rules are:

1. $\frac{\partial}{\partial \boldsymbol{B}} tr\left(\boldsymbol{BAB}'\right)=\boldsymbol{BA}'+\boldsymbol{BA}$ (TMC.111)
2. $\frac{\partial}{\partial \boldsymbol{B}} tr\left(\boldsymbol{BC}\right)=\boldsymbol{C}'$ (TMC.100)
3. $\frac{\partial}{\partial \boldsymbol{B}} tr\left(\boldsymbol{DB}'\right)=\boldsymbol{D}$ (TMC.104)
4. $\frac{\partial}{\partial \boldsymbol{B}} tr\left(\boldsymbol{G}\right)=\boldsymbol{0}$

This produces first order condition:

$$
\begin{aligned}
\boldsymbol{0}=&2\boldsymbol{B}E\{\boldsymbol{FF}'\}-E\{\boldsymbol{XF}'\}-2\boldsymbol{B}E\{\boldsymbol{F}\}E\{\boldsymbol{F}'\}+E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}-E\{\boldsymbol{XF}'\}+\boldsymbol{0}+E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}-\boldsymbol{0}-2\boldsymbol{B}E\{\boldsymbol{F}\}E\{\boldsymbol{F}'\}+E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}+E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}\\
&-\boldsymbol{0}+2\boldsymbol{B}E\{\boldsymbol{F}\}E\{\boldsymbol{F}'\}-E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}-E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}+\boldsymbol{0}\\
=&2\boldsymbol{B}\left[E\{\boldsymbol{FF}'\}-E\{\boldsymbol{F}\}E\{\boldsymbol{F}'\}\right]-2E\{\boldsymbol{XF}'\}+2E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\boldsymbol{B}\left[E\{\boldsymbol{FF}'\}-E\{\boldsymbol{F}\}E\{\boldsymbol{F}'\}\right]&=E\{\boldsymbol{XF}'\}-E\{\boldsymbol{X}\}E\{\boldsymbol{F}'\}\\
\boldsymbol{B}Cov\{\boldsymbol{F}\}&=Cov\{\boldsymbol{X},\boldsymbol{F}\}\\
\boldsymbol{B}&=Cov\{\boldsymbol{X},\boldsymbol{F}\}Cov\{\boldsymbol{F}\}^{-1}.
\end{aligned}
$$

[I don't have the energy to take second order conditions.]

7.1.1

Assume a bivariate market where the prices at the investment horizon, (P_1, P_2), have the following marginal distributions:

$$P_1 \sim \text{Ga}\left(\nu_1, \sigma_1^2\right); \tag{438}$$

$$P_2 \sim \text{LogN}\left(\mu_2, \sigma_2^2\right). \tag{439}$$

Assume that the copula is lognormal, i.e. the grades (U_1, U_2) of (P_1, P_2) have the following joint distribution (not a typo, why?):

$$\begin{pmatrix} \Phi^{-1}(U_1) \\ \Phi^{-1}(U_2) \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right); \tag{440}$$

where Φ denotes the CDF of the standard normal distribution. Assume that the current prices are $p_1 \equiv E\{P_1\}$ and $p_2 \equiv E\{P_2\}$. Fix arbitrary values for the parameters $\left(\nu_1, \sigma_1^2, \mu_2, \sigma_2^2\right)$ and compute the current prices.

1. equations 438 and 439 give the univariate marginal distributions for P_1 and P_2;
2. fully specifying the distribution requires knowing how the marginals are bound
3. we are told that they are bound by a "lognormal" copula
4. I don't know what a lognormal copula is, but can read equation 440; thus, I ignore the words, and try to understand 440
5. by Meucci (2005) equation (2.27), I can define a $Z_i \equiv Q_{Z_i}(U_i)$ for each of the grades, U_1 and U_2, where Q is any quantile function. The quantile function selected determines the distribution of Z_i.
6. as the inverse of a univariate CDF is a quantile function, we can write $Z_i = F_{Z_i}^{-1}(U_i)$; use Φ for the univariate CDF, so that $Z_i = \Phi^{-1}(U_i)$. Thus, Z_i is normally distributed.
7. 440 may now be re-written as
$$(Z_1, Z_2) \sim N\left(\mathbf{0}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$
8. as quantile functions are monotonic, the Z_i's are monotonic transformations of the grades (and vice versa)
9. thus, Proposition 5.6 from McNeil, Frey, and Embrechts (2005) applies (or Meucci (2005) equations (2.35) and (2.38)), and (U_1, U_2) has the same copula as does $\boldsymbol{Z}$
10. we may now turn to Meucci (2005) (2.176) and (2.177) for properties of the bivariate Gaussian copula

7.3.2

Consider the copula in Exercise 7.1.1, but replace the marginal distributions as follows:

$$P_1 \sim \mathrm{N}\left(\mu_1, \sigma_1^2\right); \tag{442}$$
$$P_2 \sim \mathrm{N}\left(\mu_2, \sigma_2^2\right). \tag{443}$$

Consider the case where the objective is final wealth. Consider an exponential utility function:

$$u(\psi) \equiv a - b \exp\left(-\frac{\psi}{\gamma}\right) \tag{444}$$

Compute analytically the certainty equivalent as a function of a generic allocation vector, (α_1, α_2). What is the effect of a and b?

How do we "notice that normal marginals [bound together by] a normal copula give rise to a normal joint distribution"? The trick is not to work with Sklar's theorem directly: it is stated in terms of distribution functions, and Meucci (2005) only provides the density function for the bivariate normal (Gaussian) copula.

1. Meucci (2005) (2.30) relates marginals (known), the copula (known) and the joint distribution (unknown)[1]

$$f_{\boldsymbol{U}}(u_1, u_2) = \frac{f_{\boldsymbol{P}}\left(Q_{P_1}(u_1), Q_{P_2}(u_2)\right)}{f^N_{\mu_1,\sigma_1^2}\left(Q_{P_1}(u_1)\right) \times f^N_{\mu_2,\sigma_2^2}\left(Q_{P_2}(u_2)\right)}$$

   where

   - by (2.176)

   $$f_{\boldsymbol{U}}(u_1, u_2) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{\rho^2 z_1^2 - 2\rho z_1 z_2 + \rho^2 z_2^2}{1-\rho^2}\right)$$

   where, by (2.171), $z_i \equiv \frac{Q_{P_i}(u_i)-\mu_i}{\sigma_i}$

   - by (1.67)

   $$f^N_{\mu_i,\sigma_i^2}\left(Q_{P_i}(u_i)\right) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{1}{2} z_i^2\right)$$

2. substitution and re-arrangement then yields

$$f_{\boldsymbol{P}}\left(Q_{P_1}(u_1), Q_{P_2}(u_2)\right) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)}} \exp\left(-\frac{1}{2}\frac{z_1^2 - 2\rho z_1 z_2 + z_2^2}{1-\rho^2}\right)$$

[1] Where does this equation come from?

which, by (2.170), is the JDF of the bivariate normal with location parameter $\boldsymbol{\mu}$ (why?) and, by (2.168), scatter parameter

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

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References

McNeil, Alexander J., Rüdiger Frey, and Paul Embrechts (2005). *Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton Series in Finance. Princeton University Press.

Meucci, Attilio (2005). *Risk and Asset Allocation*. Springer Finance. Springer.