Additive externality games

Colin Rowat
Department of Economics
University of Birmingham
c.rowat@bham.ac.uk
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Chapter 1

Introduction

1.1 Greenhouse gas emissions as a non-cooperative problem

The belief that human activity influences global climate has received considerable attention in recent decades. One channel of influence to be particularly examined is that involving the atmosphere's composition. The principal reasons for this interest are based on the following observations. First, gases are selectively permeable, allowing certain wavelengths of radiation to pass, while trapping others. Thus, altering the composition of a gas exposed to radiation may alter its energy retention properties. It is also known that carbon dioxide is a byproduct of the combustion of fossil fuels, and that, accordingly, atmospheric carbon dioxide levels have risen substantially from 288 ppmv in 1860 to some 368 ppmv today.[1] Furthermore, carbon dioxide is a so-called 'greenhouse gas', allowing passage of solar radiation but not heat.

These observations are neatly illustrated by the historical record, which shows a high degree of correlation between carbon dioxide concentrations and temperature. Figure 1.1 contains two 420,000 year data series, both reconstructed from ice samples drawn from the Vostok research station in the Antarctic (q.v. [JLP+87], [JBB+93], [JWM+96] and [PRB+99]). The first, plotted in thick lines, reconstructs atmospheric carbon dioxide concentrations from air samples frozen into the ice. The second, in thin lines, reconstructs a temperature series on the basis of the ice core's deuterium content. While both series are influenced by missing variables reflecting the incidence of solar radiation, graphs of this sort do summarise the perception that atmospheric carbon dioxide and temperature have been historically related.

[1] "Current Greenhouse Gas Concentrations", Carbon Dioxide Information Analysis Center, US Department of Energy. August 2000. The unit of measurement used here is 'parts per million by volume'.

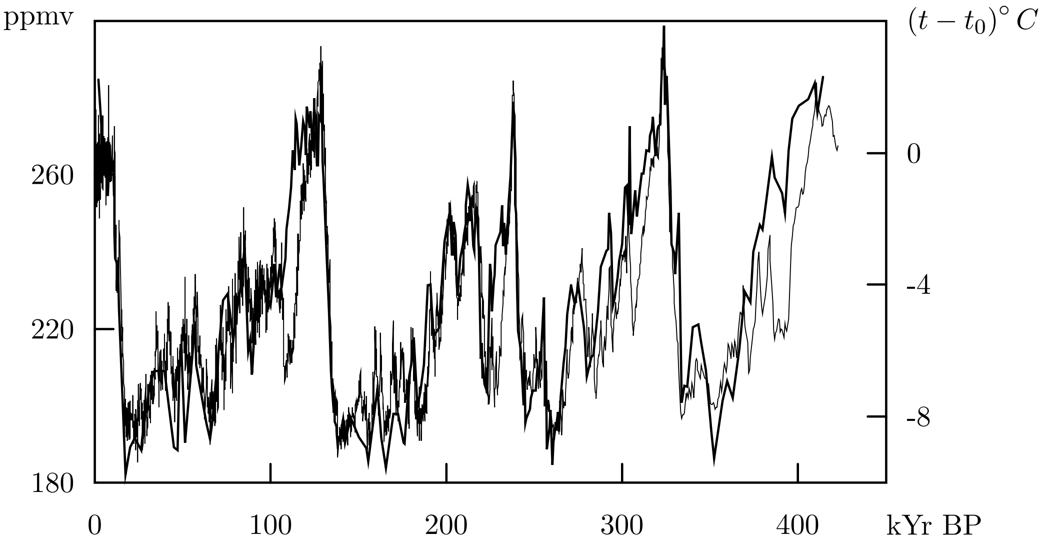

Figure 1.1: Temperature and atmospheric CO_2 against years before present

The possibility that anthropogenic greenhouse gas emissions may be influencing global climate has generated both academic and political efforts in global commons management. Perhaps the most visible manifestation of the political branch of these efforts is the Kyoto Protocol process, whereby nations are attempting to negotiate legally binding greenhouse gas emissions limits and accounting techniques. It is unclear as yet whether this attempt to exploit gains to international cooperation in the setting of greenhouse gas emissions will succeed. At root, Kyoto's social planner may be impotent: it appears to grant itself no powers of enforcement.

Under the Kyoto process, a number of industrialised countries have committed to meet emissions targets, expressed as a percentage of their 1990 emissions levels, by 2008 - 2012. Emissions targets are denominated in CO_2 equivalents, a measure that expresses the relative ability of greenhouse gases to retain heat in the atmosphere. The first column of Table 1.1 displays the 1990 greenhouse gas emissions (in gigagrams of CO_2 equivalent) of the committing countries;[2] the second column displays their 1997 emissions, the most recent data required of them [Sub99, Tables A.1, A.2]; note that not all countries have submitted this as yet, although required to do so by 1999. The third column indicates their emissions targets, as a percentage of their 1990 emissions.[3]

[2]Some Central and Eastern European countries use other years as their baselines.

[3]The European Union countries are collectively required to reduce their emissions to

Table 1.1: Greenhouse gas emissions and Kyoto Protocol commitments

Country	1990 emissions	1997 emissions	Kyoto target
Australia	474,529	n.a.	108
Austria	60,427	64,473	EU
Belgium	137,219	n.a.	EU
Bulgaria	131,436	78,609	92
Canada	555,450	663,580	94
Czech Republic	187,556	153,593	92
Denmark	70,734	82,659	EU
Estonia	29,402	31,090	92
[European Community]	n.a.	n.a.	92
Finland	72,786	76,684	EU
France	501,713	490,298	EU
Germany	1,176,328	1,012,679	EU
Greece	105,235	n.a.	EU
Hungary	98,537	n.a.	94
Iceland	2,577	n.a.	110
Ireland	51,701	n.a.	EU
Italy	508,813	n.a.	EU
Japan	1,129,359	1,280,365	94
Latvia	24,843	1,643	92
Liechtenstein	238	n.a.	92
Lithuania	42,700	n.a.	92
Luxembourg	13,153	n.a.	EU
Monaco	111	147	92
Netherlands	215,552	n.a.	EU
New Zealand	51,628	59,153	100
Norway	42,284	39,276	101
Poland	529,540	385,699	94
Portugal	67,290	n.a.	EU
Romania	261,954	n.a.	92
Russian Federation	2,648,062	n.a.	100
Slovakia	68,239	n.a.	92
Slovenia	16,919	n.a.	92
Spain	301,431	n.a.	EU
Sweden	35,099	40,656	EU
Switzerland	49,389	46,890	92
Ukraine	867,113	397,664	100
United Kingdom	748,772	674,219	EU

92% of their 1990 emissions, in an arrangement known as a bubble. The emissions restrictions agreed are contained in Annex B of the Protocol.

Country	1990 emissions	1997 emissions	Kyoto target
USA	4,841,370	5,866,128	93

At first glance, Table 1.1 is not unreassuring. Total emissions for those countries reporting in both 1990 and 1997 are 11,105,299 and 11,445,505 Gg, respectively, a 3% increase. As the Kyoto Protocol was only developed in March 1997, this slight increase may not be cause for concern. A second look at the table is more worrying if one notices the difference between the transition economies and those economies whose development in the 1990s has been less unusual. Summing the emissions of the non-transition economies reporting in both 1990 and 1997 yields 9,335,450 and 10,397,207, an 11% increase. To the extent that these economies may be exhibiting more representative development, it is possible that meeting the Kyoto targets will not be a trivial task.[4]

This concern generalises: it may be that the current arrangement of international institutions hinder holding nations to any emissions targets other than those that nations would meet non-cooperatively. The behaviour of the US Senate suggests some of the difficulties faced: it has affirmed that it will not support any US emissions targets - including those negotiated under the Kyoto mechanism - unless countries like China and India, not among the countries required to undertake emissions reductions, are also bound to reductions.[5] Recent European experience may also indicate the difficulties of reducing fossil fuel use through fuel taxes: in the face of prices regarded by consumers as high, governments come under popular pressure to reduce these taxes, which are responsible for most of the consumers' price.

The above concerns suggest the possibility that actual greenhouse gas emissions may resemble non-cooperative rather than cooperative outcomes. This possibility is sufficient to merit exploration of how non-cooperation emissions might be analysed. This thesis explores that general question by asking two specific questions. Both questions are addressed within the context of what is here called an additive externality game: a game in which an externality is generated as an additive function of agents' behaviour.

[4]Recent scientific evidence suggests that this may be easier than previously expected. A recent report argues that "rapid warming in recent decades has been driven mainly by non-CO_2 greenhouse gases" and that the "growth rate of non-CO_2 greenhouse gases has declined in the past decade" [HSR+00]. As non-CO_2 gases are less intrinsic to modern industrial processes, and apparently disproportionately significant, their reduction may be reasonably inexpensive. While these gases may offer the least expensive opportunities for reductions, it is plausible to imagine that eventually attention will return to carbon dioxide.

[5]q.v. S.Res.98, "Expressing the sense of the Senate regarding the conditions for the United States becoming a signatory to any international agreement on greenhouse gas emissions under the United Nations", 25 July 1997.

The first question asked is how emissions might evolve when nations are able to interact only via their emissions. This question is pursued by analysis of a differential game, a dynamic game in continuous time. Oligopolies and commons problems have been modelled as differential games; the techniques are equally applicable to analysis of tariff wars and the interactions of national monetary policies. They are used here to explore a simple dynamic commons problem in which agents' controls are bounded and non-linear.

The treatment given to non-linear strategies in differential games in the early 1990s has been felt by some to be unsatisfactory. In particular, the state space over which strategies have been required to hold has typically been subject to endogenous bounds. This thesis reworks that literature without this bound and then presents extended models. The general conclusion is that equilibria with non-linear strategies do not seem to be robust.

This question occupies the bulk of the thesis.

The second question asked is how emissions might evolve when nations are also able to interact by offering each other voluntary transfers. The debate over the ability to 'trade' emissions under the Kyoto process may be interpreted as a debate over the ability to offer transfers. The practical motivation for such a question is thus obvious. It coincides with a theoretical interest in whether 'non-cooperative cooperation' can improve outcomes in an externalities problem. This approach, of expanding agents' strategy spaces, contrasts with the usual approaches to externality problems, such as Pigovian taxes and Coasian property rights, by not relying on a social planner.

As adding the second dimension to each agent's strategy space increases the problem's complexity, it is modelled as a one-shot game. While the first part of the thesis looks for subgame perfect Nash equilibria, the Nash concept must be modified when agents take action in an attempt to influence each other. This part therefore uses a functional Nash equilibrium.

The finding in this section of the thesis is that self-enforcing transfers can be Pareto improving, even if they are symmetric, thereby netting out in equilibrium. It is unclear whether this is surprising or expected. In a single agent control problem, an additional control is certainly expected to not worsen payoffs. In a game, though, an additional control may also reduce the ability of agents to commit themselves, possibly causing the outcome to further deviate from the Pareto optimal.

Although the transfers are required to be self-enforcing, this approach does introduce cooperative elements, in violation of the spirit of this thesis: notional stages are introduced, implying a commitment to a particular game form; it must also be assumed that a transfer function offered will be adhered to when the time comes to make payments.

The conceptual problems of how nations might behave cooperatively in the international realm might be overcome if nations are allowed to enact and enforce national laws that bind their behaviour. This possibility makes this thesis' commitment to non-cooperative interaction seem rather rigid.

The questions explored here have, of course, been explored before. This introductory chapter therefore discusses various approaches to them and related problems that the existing literature has pursued. It then introduces at greater length the present approach and results. These fall far short of offering a sophisticated understanding of greenhouse gas emissions. That would require a model of heterogenous agents with evolving characteristics, uncertainty of preferences and a stochastic evolution of the state of nature. Hopefully, though, some more foundation stones have been laid that might assist in future analyses of this and related questions.

1.2 Existing approaches and literature

As any modelling exercise, attempting to formulate greenhouse gas emissions scenarios is an exercise in the judicious choice of lies. Before introducing the set of lies used in this thesis, some of the existing ones are presented. Some of these are more applicable to this thesis' first question (no cooperation) while others are more applicable to the second (non-cooperative cooperation).

1.2.1 Social planners & cooperative solutions

A standard approach to commons problems has a social planner calculate and implement a cooperative outcome based on some relative weightings of countries' importances. One drawback of this approach in the context of greenhouse gas emissions is that it is not clear that there is a social planner capable of implementing an international environmental agreement.

1.2.2 One shot cartel games

There is a large literature on one-shot cartel games (q.v. [Car97]). This literature typically examines n identical agents playing a game with externalities. Agents choose between joining a cartel or remaining part of a competitive fringe. Those joining the cartel maximise its joint profits against the non-cartel members' actions; those on the competitive fringe individually profit maximise.

This literature then attempts to identify the stable cartel structures. Typically, two profit functions are plotted as a function of the number of cartel members: cartel membership profits and fringe membership profits. A cartel is stable if no cartel member prefers to join the fringe (the gains from individually profit maximising being offset by the reduction in price due to the loss of cartel power) and vice versa. Different techniques for enlarging the stable cartel may then be explored (e.g. playing the one-shot game repeatedly).

An obvious drawback of this approach to problems involving persistent goods or bads is that it does not model state variables. Nevertheless, Carraro and Siniscalco, two contributors to this literature, believe that cartel games are of interest. They argue that, in the case of GHG emissions, "the level of emissions, however, can hardly be conceived as a trigger variable which can be increased strategically in response to other countries' defection" [CS93, p. 312]. Dockner [Doc92] has provided a more rigourous link between the steady state closed-loop equilibria of dynamic models and the 'conjectural variations' equilibria of one-shot static models. A computational example of this literature is found in Babiker [Bab97].

1.2.3 Private provision of a public good

In the private provision of public goods (or bads) literature, agents typically take turns making contributions to an exogenously defined 'project' which provides a public payoff stream upon completion (q.v. Admati and Perry [AP91] for a two agent alternating order game with complete information, Gradstein [Gra92] for one with incomplete information). Variants on this theme have payoffs accruing while the project is still under construction (q.v. Marx and Matthews [MM97]) or require a single agent to assume a project's whole burden (q.v. Bilodeau and Slivinski [BS96]).

Solutions are usually derived by backward induction. After some sequence of contributions to a project, an impatient agent will prefer to pay to finish the project now rather than not paying but wait for the next agent to do so at the next point in (discrete) time. Identifying the amount that an impatient agent will pay now to finish the project then allows consideration of the sub-problem which has a project of the original size less what the impatient agent would pay now. Backward induction then continues.

The exogenous definition of projects is a weakness in the non-cooperative context. Their endogenisation requires some mechanism for proposing and accepting or rejecting projects (e.g. agents take turns proposing projects until all accept one and its variants, such as first rejector proposes; q.v. [AP91]; Chatterjee *et al.* [CDRS93] for a fixed order with n agents or Okada [Oka96] for a random order). As adherence to any such mechanism requires cooperation, this approach compromises a non-cooperative approach.[6]

1.2.4 Time series analysis

A very different attempt to generate emissions scenarios is to take historical data, fit a time series model to them and then predict out of sample. One drawback of this approach is that the distance out of sample that one is interested in may be quite long relative to the sample. Furthermore, there is

[6] For an illustration of the problem of agreeing to rules, see the chess game story in Marquez [Gar84].

the possibility of a structural break: countries have not historically regarded greenhouse gas emissions as strategic. If they do in the future then a model calibrated on past behaviour may not be useful.

1.2.5 Optimal growth: RICE/DICE

Nordhaus and Yang's Regional Integrated Model of Climate and the Economy (RICE) model is the computational benchmark in the climate games literature [NY96]. It turns the intertemporal optimisation problem of Nordhaus' earlier DICE (replace "Regional" with "Dynamic") into a multi-agent game.

The original DICE is a discrete time, finite horizon Ramsey optimal growth problem, augmented by environmental components. These include a new choice variable (rate of emissions reduction) and equations for the environment as well as its interaction with the economy. RICE extends DICE by breaking the world into five agents.

Nordhaus and Yang calculate three solutions to their RICE model. The first has agents optimising heedless of the externalities. The second is a social planner's optimum. The third is a Nash equilibrium in open loop strategies (as noted by Radner [Rad98, p. 10]). Thus agents choose an optimal timepath of controls, assuming others' to be fixed. Nordhaus and Yang find a unique fixed point by inducting backwards over a finite horizon.

While RICE's spirit - a non-cooperative dynamic game - is shared with the present approach, it also differs in a number of respects. RICE has stayed closer to the motivating question of greenhouse gas emissions. Its model of the economy, the environment and their interactions is richer than that in this thesis. In return, though, it works with discrete time, a finite horizon and open loop strategies. In contrast, the first section of this thesis works with continuous time, an infinite horizon model and closed loop strategies.

1.2.6 Scenario writing

Given the complexity of the environment-economy system some analysts have developed scenarios on the basis of qualitative assessments. Key variables are identified and their timepaths fixed by various assumptions. Each set of assumptions is then used to seed a quantitative model, generating scenarios.

The UN Intergovernmental Panel on Climate Change (IPCC) has used this approach. Their recent Special Report on Emissions Scenarios starts with four stories (deliberately omitting the possibility of disasters), from which 40 scenarios are generated [Int00]. This report does not assign probabilities to the scenarios.

1.2.7 Bargaining models

Rubinstein bargaining games and their generalisations might help explore 'non-cooperative cooperation'. Burbidge *et al.* identify some weaknesses with this approach:

> But any extensive form one writes down as a description of the protocol of dynamic negotiation is bound to be somewhat arbitrary; and, as is well-known, equilibria in dynamic games are often extremely sensitive to the precise protocols. [BDMS97, p.953]

A second difficulty was already encountered in the context of private provision games: it is a strong assumption to believe that agents playing non-cooperatively might nevertheless agree to abide by an arbitrary extensive form.

1.2.8 Coalitional models

Another approach to 'non-cooperative cooperation' is offered by the literature on coalitions. This literature, though, has many of its own difficulties. First, many of the models assume transferable utility, avoiding the problem of ordinal utility. Second, the threat point that agent i can offer is usually some version of the singleton $\{i\}$ rather than an alternative coalition. Consideration of possible alternative coalitions is, as usual, more complicated but more sensible. A third potentially unsatisfactory feature of the coalitional approach is the method used to share the surplus available to a coalition. This are usually divided according to some co-operative rule (e.g. Shapley or CS-value, q.v. [HK83]); sometimes agents are allowed to make irrevocable commitments to the resulting coalitions and the sharing rules (q.v. [BDMS97]). One wonders what new technology has allowed this to happen in an otherwise non-cooperative environment.[7]

Bloch [Blo00] presents a good survey of the coalitional literature.

1.2.9 Menu auction and common agency

Another approach to co-operation and collusion may be found in the menu auction and common agency literatures. These were pioneered by two 1986 papers by Bernheim and Whinston (q.v. [BW86b] and [BW86a]). In these, multiple principals provide reward schedules for a single agent. The agent reacts accordingly so that the equilibrium concept resembles that of the Stackelberg leader-follower model.

[7] A related question is that of how the coalition then plays with non-coalition members. Both questions, that of surplus division and that of the coalitions' play, are questions about the objective function of the coalition.

This approach does not seem appropriate in this thesis as the context makes exogenous third party agents difficult to justify. Trying to endogenise the agent might be a promising research programme as there does not yet seem to be a well developed story of how social planners arise.

1.3 Current approach and results

This section presents the assumptions and techniques used in this thesis. Results are also introduced.

1.3.1 Assumptions

Throughout this thesis, agents are regarded as rational in the standard game theoretic sense. This clearly implies stronger assumptions about their abilities than would, for example, models of boundedly rational agents. As the agents here represent governments, though, this may be a less implausible assumption than it would be were they, for example, individuals.

Relatedly, the intertemporal models developed here assume agents to optimise over an infinite horizon. This assumption again makes computational demands on the agents which would be harder to sustain were they not national governments.

Another assumption made throughout is that the agents are governments. To an economist, this may seem quite natural. At the same time, there is a common popular perception that the nation state has declined in influence relative to large corporations. Should not corporate agency, then, be modelled? The present assumption that governments are agents says nothing about the aggregation of preferences within a country. It merely contends that a nation's action can be taken to reflect an aggregation of some sort, influenced either passively or actively by the government. It makes no assumptions on the weights assigned to the interests aggregated.

1.3.2 The differential game models

This thesis' first question has been modelled with a differential game principally because that seemed to stay closest to a non-cooperative motivation. Private provision approaches were initially pursued but the problem of determining which project agents would seek to implement led, in the end, to their abandonment.

Introduction to differential games

A differential game is a multi-agent control problem in continuous time. Each agent typically controls a single real-valued control variable, influencing a vector of real-valued state variables. If controls are bounded, as here, the most common bound is their restriction to the non-negative reals. The term differential is used as the state variables' equations of motion are differential equations.

Much of the existing literature works with a scalar state variable (cf. Mason [Mas] and the references therein for exceptions). In some cases there is only one state variable of interest but this simplification also facilitates analysis by reducing the Hamilton-Jacobi-Bellman equations from partial to ordinary differential equations.

A continuous time formulation brings advantages and disadvantages. There are at least two advantages in this context. First, it is a more plausible formulation for a problem in real time. Second, it prevents implicit periods of commitment being built into the model, consistent with this thesis' commitment to non-cooperative models.

A feature of continuous time models frequently regarded as disadvantageous is the possible non-uniqueness of the state variable's timepath. This is typically addressed by requiring strategies to be Lipschitz continuous but there are also weaker ways of ensuring uniqueness.[8] While the plausibility of continuity may be defended on the grounds that the control variable may be 'sticky' it may be preferable, were this the case, to model its 'stickiness' directly.

The usual solution concept in a differential game is the Nash equilibrium. In dynamic games a refinement of the Nash, the Markov perfect equilibrium (MPE), requires that solutions be functions of the state variables alone (hence Markov) and that they be Nash equilibria of every subgame (hence perfect).[9] This restriction then allows each agent's problem to be treated as an optimal control problem against fixed play by the others.

The Markov approach therefore generally omits a history of play. In the optimal control problem, this is irrelevant: there is no history of strategic interaction. In the game, the omission may have consequences. The infinitely repeated prisoners' dilemma is the standard example: conditioning strategies on the null state variable alone causes an infinite repetition of the static game's Nash equilibrium as deviation cannot be punished. The use of a state that insufficiently summarises the game's history may therefore paint a worst case scenario of cooperation.

Nevertheless, MPE are still perfect equilibria in broader classes of strate-

[8] There are papers in the differential game literature, such as Dockner and Sorger's [DS96] fishing game, that find discontinuous Markov strategies. Their approach, less mechanical than that presented here, is made possible by their choice of functional form. Sorger [Sor98] generalises their results beyond two players. Dutta and Sundaram [DS93], in a variant on Levhari and Mirman's [LM80] discrete-time classic, avoid tragedy by observing that punishments triggered by the state variable may proxy those triggerred by play; their paper proves that strategy continuity and strategy monotonicity are both sufficient conditions for a tragedy. These papers, though, are the exception.

[9] The literature often refers to 'feedback', 'memoryless closed loop' and 'Markov' strategies interchangeably.

gies as long as the state contains all payoff relevant variables.[10] For attempts to derive endogenously the payoff relevant state variables see Maskin and Tirole [MT97]. For an example (although not in Markov strategies) of an attempt to bridge the gap between the small state used by practitioners and theorists' larger sufficient state see Kubler and Schmedders [KS99].

Together, the usual continuity requirement on strategies and the removal of past play information from the state variable impede play-dependent trigger strategies. These assumptions may therefore be responsible for some of this literature's 'tragedy' results, in which stocks are overexploited relative to the first best.

A differential game is linear-quadratic (LQG) if its equations of motion are linear in the controls and the states and if the payoff or loss functions are quadratic in the controls and states. LQG often yield closed form MPEs in strategies that depend linearly on the state; their derivation involves solving a system of coupled algebraic Riccati equations (CARE) (q.v. Başar and Olsder [BO99]). Both Başar and Olsder [BO99, p.324] and Lockwood [Loc96] present uniqueness results for linear strategies. To exploit this appealing feature of LQG, problems are often modelled as LQG, a decision defended by the argument that the LQG may be thought of as a local Taylor approximation to the actual game (q.v. Fudenberg and Tirole [FT91, p.523]).

Within the economics literature, differential game theory has been used to analyse common resource management problems (q.v. Dockner and Long [DL93] for a LQG greenhouse gas example or Dockner and Sorger [DS96] for a non-LQG fisheries example) and oligopolistic pricing problems (q.v. Tsutsui and Mino [TM90] for a LQG duopoly example). Common to these analyses are the assumptions that agents (usually two for convenience's sake) are identical, that they play identically and that the state is a scalar. The second assumption follows naturally from the first; with the third, analysis is reduced to the solution of a single ordinary differential equation and a transversality condition. Mason's acid rain game [Mas] is again exceptional as its agents may be asymmetric and it possesses a state vector.

The four examples cited in the preceding paragraph are unusual in their interest in non-linear strategies; by contrast, the bulk of the differential games literature has concentrated on linear strategies. The differential games here also consider the more general strategy space: although a restriction to linear strategies simplifies analysis, it is not clear why sophisticated agents would *a priori* restrict themselves in this manner.[11]

[10] A variable is payoff relevant if it is an argument in the instantaneous payoff function. The claim follows: as Markov strategies already condition on everything payoff relevant, there is no unilateral incentive to condition on anything more.

[11] A posterior rationale for restriction might exist if multiple equilibria, including at least one in linear strategies, are found and if a linear strategy is Pareto superior to the others. When agents are patient, Dockner and Long conclude that the opposite is true:

The solution to differential games with non-linear strategies no longer centres on solving the CARE system: Dockner and Sorger solve their ordinary differential equation without further differentiation while Tsutsui and Mino solve theirs by differentiating the Bellman equation; the equation that they thus derive they call an auxiliary equation. Dockner and Long follow Tsutsui and Mino, altering the payoff function to transform the duopoly problem into a greenhouse gas one. Mason addresses his more complicated system of partial differential equations by restricting analysis to behaviour in the steady state; he uses numerical methods to solve the resulting system of equations. All four papers, as well as Sorger's generalisation of his work with Dockner [Sor98], find a continuum of equilibria.[12]

Structure and results

Chapters 2, 3 and 4 develop the thesis' first question. The basic differential game is set out in Chapter 2: identical and certain agents play a LQG without side payments; they may use non-linear strategies. Chapter 2 therefore reworks Tsutsui and Mino [TM90] and Dockner and Long [DL93], with two differences: a correction and a cosmetic change. The correction is significant. Tsutsui and Mino's equilibrium concept only requires strategies to be optimal against play within an endogenously bounded subset of the state space. By contrast, the present analyses do not endogenously bound the state space. This modifies the earlier results: for certain parameter values, the continuum of non-linear equilibria and single linear equilibrium are still found. Otherwise only the linear candidate survives, a new result. The cosmetic change involves a slightly different specification of the payoff functions; this is cosmetic as they are, both in Tsutsui and Mino and here, quadratic loss functions.

Chapter 3 extends the model of Chapter 2 by allowing symmetric agents to play asymmetrically. This is not as intrinsically interesting as a situation in which asymmetric agents play asymmetrically but its consideration has two advantages. First, it allows the results derived to be tested against the benchmarks established in Chapter 2. Second, as the model now involves solving two non-linear ordinary differential equations, it requires the same tools as would one in which the agents were asymmetric. The technique is therefore applicable to the fully asymmetric situation, but slightly more transparent.

As analytical solutions are not found to the two ordinary differential equations, numerical analysis is used. This explores both families of similar neighbouring strategies as well as the isolated strategies felt to be candi-

non-linear paths outperform the linear [DL93, p.24].

[12] It is somewhat perplexing that Başar and Olsder [BO99, Remark 6.16] claim that the question of whether there might be non-linear solutions to the LQG remains unresolved. They may have in mind similar concerns to those presented below.

dates for supporting equilibria. No new equilibria are found. As the search seems reasonably exhaustive, it seems quite likely that there are, in fact, no asymmetric equilibria. Chapter 4 uses this approach to a examine a case of asymmetric agents. The only equilibrium found is the linear.

The strong suggestion of these chapters is that the non-linear equilibria originally discovered by Tsutsui and Mino are not robust in the linear quadratic game.

1.3.3 The functional Nash equilibrium model

As above, a desire to remain close to a non-cooperative motivation influenced model choice when addressing this thesis' second question. Self-enforcing transfers seemed more appropriate from this point of view than coalitional models, with their cooperative sharing rules. The functional Nash equilibrium concept then followed as transfer strategies violate the premise of the standard Nash equilibrium, namely that agents optimise assuming the others' behaviour to be fixed.

Introduction to functional Nash equilibria

Klemperer and Meyer's 1989 paper provided a means of analysing equilibria when agents choose functions [KM89]. Their motivating example was an oligopoly problem with demand shocks that occur after firms choose their strategies. Firms therefore select functions to respond flexibly to the uncertainty. This does not require any sort of commitment technology as the functions are chosen so that any outcome is *ex post* optimal.

In Green and Newbery's study of competition in the British electricity generation industry the use of functions is even more compelling as British firms are legally required to submit supply functions the day before generation [GN92]. A real auctioneer then equates supply and demand. Their paper follows Klemperer and Meyer's closely. In Green and Newbery's paper, though, demand is a function of price and time, while in Klemperer and Meyer's it is a function of price and a shock.

Klemperer and Meyer noted that earlier work in functional strategies had not involved uncertainty, giving rise to at least two problems. The first of these is the motivational one mentioned above: the flexibility gained by choosing functions is less sensible in a perfectly predictable environment. The second problem is that the set of functional equilibria expands considerably without uncertainty.

To explain this second problem, consider two duopolists deciding on their production levels of a homogeneous good. As the firms have preferences over both production decisions, almost all points in (q_1, q_2) space may be described as the intersection of indifference curves of firms one and two. There are then two sources of multiplicity. First, given any $\mathbf{q}^* = (q_1, q_2)$

at which indifference curves intersect, the firms can choose supply functions such that the other's best response produces an outcome at $\mathbf{q}^*$. Second, as very few conditions are placed on the supply functions supporting $\mathbf{q}^*$ (slope at $\mathbf{q}^*$ and possibly properties designed to prevent a second intersection), a multiplicity of functions will be able to support an equilibrium at any $\mathbf{q}^*$ if non-linear strategies are permitted.

The introduction of demand uncertainty by Klemperer and Meyer imposes more conditions on the supply functions by requiring that optimality hold for all possible realisations of the uncertainty. Consequently, as the support of the uncertainty increases, so does the range over which these conditions must hold. This, in their model, serves to refine the equilibrium set.

Structure and results

Chapter 5 asks how much improvement is possible with voluntary transfers. It analyses two games in depth, a benchmark in which transfer functions are not allowed and an expanded game in which they are. In the first case, there are two equilibria in linear emissions. In the second, there are again two equilibria when emissions strategies are restricted to being linear and transfers quadratic functions. None of the game results allow attainment of the first best. Furthermore, neither of the games is Pareto superior in the sense that either of its outcomes are preferred to both of the other. In both games, the equilibrium strategy supported by the emissions function that decreases in the state variable is preferred to that supported by the increasing emissions function.

Chapter 2

Two symmetric agents, symmetric play

2.1 Introduction

In this chapter two identical and static agents emit greenhouse gases into a common atmosphere as a side effect of production, their single control variable. Only production and the greenhouse gas stock are payoff relevant, the latter as a wealth effect. Agents' strategies are (nonlinear) Markov. The model analysed here is therefore very similar to that of Tsutsui and Mino [TM90]. This chapter differs from their paper in at least two ways.

The chief difference lies in the domain over which strategies must be defined, and their performance assessed. Tsutsui and Mino endogenise the domain as follows. The differential equation that they derive from their Bellman equation produces an infinite number of solutions, parameterised by a constant. For each individual solution to the differential equation, they define a domain such that, within that domain, the solution satisfies the control and state bounds and forms a well-defined $\mathcal{C}^1$ function. These individual domains, over which behaviour is evaluated, are generally smaller than $\Re_{++}$, their original state space.

This endogenisation both seems unnatural and has serious implications: for a particular strategy to support a Nash equilibrium it must be that the agent regards that strategy as yielding a superior payoff to any other admissible strategy. Payoffs can only be compared, though, if agents are allowed to consider all possible strategies of play, including those strategies which would cause the state variable to leave the endogenised domain, while remaining in $\Re_{++}$. By preventing these being considered, the Tsutsui and Mino approach rules out the sort of conjecture that underlies the Nash concept.

Nevertheless, this endogenisation seems an important element of the existing literature's approach. Tsutsui and Mino write that, "the domain of the state plays a crucial role in characterising a stationary Markov feedback equilibrium" [TM90, p.140] while Dockner and Long, following them, stress, "the local nature of the nonlinear Markov strategies" [DL93, p.23], referring to the truncation of the rest of the state space.

In contrast, the present thesis works primarily with $Z \in \Re_{++}$; an appendix considers a subset of $\Re_{++}$. Interestingly, this consideration of the full domain throughout does not necessarily alter the equilibrium set. Under certain conditions, though, the continuum of [TM90] and [DL93] may be refined to a singleton, a downward sloping linear MPE strategy. As mentioned on page 12, one of the appeals of linear-quadratic models, such as the present, is that they often do admit such linear solution strategies.[1]

The second difference between this chapter and the existing literature is cosmetic. The instantaneous payoff functions of Tsutsui and Mino and Dockner and Long are

$$\begin{bmatrix} z & x_i & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\alpha \\ 0 & -\frac{1}{2}\alpha & 0 \end{bmatrix} \begin{bmatrix} z \\ x_i \\ 1 \end{bmatrix}; \text{ and}$$

$$\begin{bmatrix} z & x_i & 1 \end{bmatrix} \begin{bmatrix} -\frac{\gamma}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\beta}{2} \\ 0 & \frac{\beta}{2} & 0 \end{bmatrix} \begin{bmatrix} z \\ x_i \\ 1 \end{bmatrix};$$

respectively, where Greek letters denote strictly positive constants, x_i agent i's control variable and z the state variable. The instantaneous payoff function used here is

$$\begin{bmatrix} z & x_i & 1 \end{bmatrix} \begin{bmatrix} -\nu & 0 & \nu\zeta \\ 0 & -1 & \xi \\ \nu\zeta & \xi & -\xi^2 - \nu\zeta^2 \end{bmatrix} \begin{bmatrix} z \\ x_i \\ 1 \end{bmatrix}.$$

Therefore, the present work models an additive quadratic loss function in which the static optimal level of the state (ζ here) may be greater than zero. Dockner and Long assume that the static optimal state is zero.[2] Otherwise, the models are not nested versions of each other: as the first and third matrices are singular, no transformations of variables cause them to equal the second (which is only singular if $\gamma = 0$ or $\beta = 0$). This is not an important difference as the games are still linear-quadratic, and the solutions of the differential equations similar in form.

The basic linear-quadratic game is presented and analysed in Section 2.2; Section 2.3 extends the analysis to a nonlinear equation of motion. Section 2.4 concludes with some unresolved questions. Appendix A presents two

[1] See Dutta [Dut95] for an introduction to the question of whether Folk Theorem results might hold in dynamic games.

[2] Although not complicating analysis, this is not simply a normalisation as the differential equation of motion will be seen to have a decay term.

standard conditions for candidate value functions. Appendix B considers an alternative specification to the initial quadratic utility function. Finally, Appendix C considers a particular case of asymmetric play.

2.2 A linear-quadratic model

Consider two agents, indexed $i = 1, 2$. Grant them identical quadratic instantaneous utility functions[3]

$$u(x_i, z) = -(x_i - \xi)^2 - \nu (z - \zeta)^2; \tag{2.1}$$

where $x_i \in \Re_+$ are their controls, $z \in Z = \Re_{++}$ is the state and Greek letters denote strictly positive constants. The first strict inequality simplifies consideration of activity around the state space's lower bound. In the context of the greenhouse gas emissions problem, the x_i represent national production (which incidentally produces emissions in a fixed ratio to output), and the z represent the consequent atmospheric stock of greenhouse gases.

Instantaneous utility is therefore concave in both control and state. Furthermore, agents have a production glut point ($x = \xi$) and a climate glut point ($z = \zeta$).[4] The former may be consistent with an aggregated neoclassical labour supply trade off between work and leisure or an optimal capacity utilisation ratio. The latter allows agents to have some sense of optimal climate including, but not necessarily, the lunar climate, $\zeta = 0$.

Agents' intertemporal objective functions are of the form

$$\int_0^\infty e^{-\delta t} u(x_i(t), z(t))\, dt; \tag{2.2}$$

where $\delta \in \Re_{++}$ is a discount rate and t is time, assumed, for calibration purposes, to be in years.

The evolution of GHG stocks is described by the linear differential equation of motion

$$\dot{z} = x_1(t) + x_2(t) - \beta z(t) \ \text{ s.t. } \ z(0) = z > 0; \tag{2.3}$$

where β is a constant decay rate; this last component of the equation of motion is often called the *assimilation function*. Some specifications of this function include a constant negative term; that is not done here for simplicity's sake as it would require an auxiliary condition to prevent $\dot{z}(0) < 0$ when $x_1 = x_2 = 0$. In section 2.3, below, the more realistic $\beta(z)$ is considered.

[3]The tools developed here apply as easily to the case in which δ, ν, ξ and ζ differ across agents.

[4]Meade's language of glut is used instead of Ramsey's of bliss: although marginal utility with respect to a factor may have reached zero, "people would not necessarily be blissfully happy; that depends upon many factors other than the economic. But economic advance would have no further contribution to make to human welfare. Call this position one of 'product glut'."[Mea55, p.94]

The linear equation of motion and quadratic objective function together define a *linear-quadratic game* (LQG). If strategies are restricted to linear functions of the state space, then linear equilibrium strategies may often be derived from the ensuing Riccati equations. As these are well known (again, see page 12 for references), the following concentrates on finding equilibria in non-linear strategies.

2.2.1 Some reference non-game payoffs

The glut point

The payoff to being at the glut point, $(z, x_i) = (\zeta, \xi)$, forever is zero. While this is not attainable as a steady state except when $\beta\zeta = 2\xi$, it does impose a finite upper bound on payoffs. It is therefore necessary that any solution to this problem have a payoff that is bounded above by zero.

The first best

The steady state of the first best when agents are identical and equally weighted by the social planner may also be calculated. The first order necessary conditions of the current value Hamiltonian are

$$\begin{aligned} m(t) &= -u_x(x_i, z); \\ \dot{m}(t) - (\beta + \delta) m(t) &= 4\nu (z - \zeta); \end{aligned}$$

where m is the current value Lagrangian multiplier and u_x is the partial derivative of $u()$ with respect to its first argument. These imply a system of differential equations.[5] Rather than solving the full timepaths note that, in the steady state, $x_i(t) = \frac{\beta}{2} z(t)$ and $m(t) = -4\nu \frac{z-\zeta}{\beta+\delta}$. Combined with the first order conditions these yield the steady state

$$(z_{fb}, x_{fb}) = \left(2\frac{(\beta+\delta)\xi + 2\nu\zeta}{\beta(\beta+\delta) + 4\nu}, \beta\frac{(\beta+\delta)\xi + 2\nu\zeta}{\beta(\beta+\delta) + 4\nu} \right). \tag{2.4}$$

When $\beta\zeta \leq 2\xi$, this exceeds the climate glut level and falls below the product glut level. As this is the first best, though, it is optimal by definition and cannot be considered a 'tragedy' result.

As the relationship between β, ξ and ζ encountered in the previous paragraph recurs throughout the chapter it is formalised as a domain restriction on β:

[5] The dynamic programming approach does not give any clearer an expression for the dynamics. Its differential equation,

$$w'(z) = \frac{(\beta+\delta) w(z) + 4\nu (z - \zeta)}{2\xi - \beta z + w}$$

(where $w(z)$ is the derivative of the candidate value function, $W(z)$, and subscripts index agents) has an unwieldy implicit solution.

A1 $\beta\zeta \leq 2\xi$.

This assumption is of no analytical importance but simplifies exposition by limiting it to cases in which the steady state locus, $\dot{z}(t) = 0$, passes below the glut point, (ζ, ξ). This assumption is maintained for the remainder of this section; those cases in which it does not hold are considered are then considered.

2.2.2 Playing the game

Generally, a strategy is a rule for determining x_i at any point in time, t, as a function of t, and the history of the game, $\{x_1(\tau), x_2(\tau), z(\tau) \mid \tau \in [0, t]\}$. A *(stationary) strong Markov strategy* is a function of the current state alone. While the payoff-relevant state space may be very large it is here restricted to the state space to Z. Therefore a stationary strong Markov strategy is a mapping, $x_i : \Re_{++} \mapsto \Re_+$, so that x_i is $x_i(z)$.

A common concern expressed about work in continuous time is that differential equations may give rise to non-unique solutions. A strategy pair (x_1, x_2) is therefore *admissible* if it yields a unique, absolutely continuous solution to equation of motion, 2.3. As the solution to equation 2.3 is

$$z(t) = e^{-\beta t}\left\{z + \int_0^t e^{\beta s}\left[x_1(s) + x_2(s)\right] ds\right\};$$

this merely requires that the $x_i(t)$ be integrable. The requirement of absolute continuity simply requires that it be possible to integrate equation of motion 2.3, an obvious necessary condition for a solution.

Let $J_i(x_i|x_j, z)$ be the total payoff received by agent i when playing strategy x_i against strategy $x_j, j \neq i \in \{1, 2\}$ and starting at $z(0) = z$. Then:

Definition 2.1 *An admissible strategy pair* (x_1, x_2) *is a* Nash equilibrium *if*

$$J_i(x_i|x_j, z) \geq J_i(\hat{x}_i|x_j, z)\, \forall i \neq j \in \{1, 2\};$$

where $\hat{x}_i$ *is any other admissible strategy available to agent* i.

Definition 2.2 *A* (strong) Markov Perfect Equilibrium *(MPE) is a Nash equilibrium in (stationary) Markov strategies.*

Definition 2.3 *Let the* value of the game *to agent* i *of the game played by agents* i *and* j *be*

$$V_i(z) = \max_{x_i \geq 0} J_i(x_i|x_j, z).$$

Attention is restricted to those solutions in which $V_i(\cdot)$ is almost everywhere continuously differentiable. This allows work with the continuous version of Bellman's equation.[6] Therefore assume throughout that:

C1 $V_i(\cdot)$ is piecewise $\mathcal{C}^1$.

The approach of Tsutsui and Mino [TM90] and Dockner and Long [DL93] requires the stronger assumption that $V_i(\cdot) \in \mathcal{C}^2$; this will be seen to follow automatically at most points when **C1** holds. Against this, Dockner and Sorger [DS96] allow a discontinuous value function; they then derive MPE strategies which jump but preserve the continuity of $V_i(\cdot)$. Başar and Olsder's example 5.2 [BO99, ex 5.2, ch 8] demonstrates that value function continuity may fail even in single agent optimisation problems; the optimal control in their example follows a bang-bang pattern.

When agent 1's value function is differentiable it solves Bellman's equation:

$$\delta V_1(z) = \max_{x_1 \geq 0} \left[-\xi^2 - \nu (z - \zeta)^2 + V'(z)(x_2^* - \beta z) - x_1^2 + x_1 \left(2\xi + V'(z) \right) \right]. \tag{2.5}$$

If situations in which the value function is not differentiable are encountered it might be possible to progress by use of the left and right hand limits.

As the equation of motion makes it impossible that $z(t) = 0$ if $z > 0$ and as $\bar{z} = \infty$, no constraints are imposed on the state space in equation 2.5.

The non-negativity requirement on x_1 provides a first order necessary condition for the optimal control:

$$x_1^* \equiv \max \left\{ 0, \xi + \frac{V'(z)}{2} \right\}. \tag{2.6}$$

As equation 2.5 is concave in x_1, x_1^* is unique and a maximiser. Although x_1^* is unique, solutions to differential equation 2.5 will not be as they introduce a constant of integration.[7]

2.2.3 The differential equation

Substitute the conditions of equation 2.6 into the Bellman equation 2.5. As the differential equation generated produces a family of solutions, denote the family of *candidate value functions* so generated by $\mathcal{W}$; an individual member of that family is referred to as W. Therefore $V_1 \in \mathcal{W}$. Substitute $x_1^* = x_2^* = \max \left\{ 0, \xi + \frac{V'(z)}{2} \right\}$ into the Bellman equation to obtain

[6] In the differential games literature this is often referred to as the Hamilton-Jacobi-Bellman (HJB) equation.

[7] This is the case even when the shape of the value function is known as in, for example, the case of a linear strategy, $x(z) = az + b$ (so that $V''(z) = 2a$) or Dockner and Sorger [DS96] (strictly concave).

$$\delta W(z) = \left\{ \begin{array}{l} -\nu(z-\zeta)^2 + W'(z)(2\xi - \beta z) + \frac{3W'(z)^2}{4}, W'(z) \geq -2\xi \\ -\xi^2 - \nu(z-\zeta)^2 - \beta z W'(z), W'(z) \leq -2\xi \end{array} \right\} \tag{2.7}$$

Symmetric play has now been imposed. The remainder of the analysis of this problem may be broken into two steps. The first, and easier, solves the two terms of equation 2.7; this occupies the next subsections. The more difficult step involves refining $\mathcal{W}$ in an attempt to identify constants of integration consistent with the requirements of optimal play's value function. Equation 2.7 and its solutions therefore play an important role throughout this chapter.

Corner solutions

The solution to equation 2.7 when $W'(z) \leq -2\xi$ is

$$W(z) = -\frac{\xi^2 + \nu\zeta^2}{\delta} - \frac{\nu}{2\beta+\delta}z^2 + \frac{2\nu\zeta}{\beta+\delta}z + Cz^{-\frac{\delta}{\beta}};$$

where C is a constant of integration. The condition on $W'(z)$ only allows this to hold for values of z satisfying

$$\frac{2\beta}{\delta}\left[\xi + \nu\frac{\zeta}{\beta+\delta}\right]z^{1+\frac{\delta}{\beta}} - \frac{2\beta\nu}{(2\beta+\delta)\delta}z^{2+\frac{\delta}{\beta}} \leq C. \tag{2.8}$$

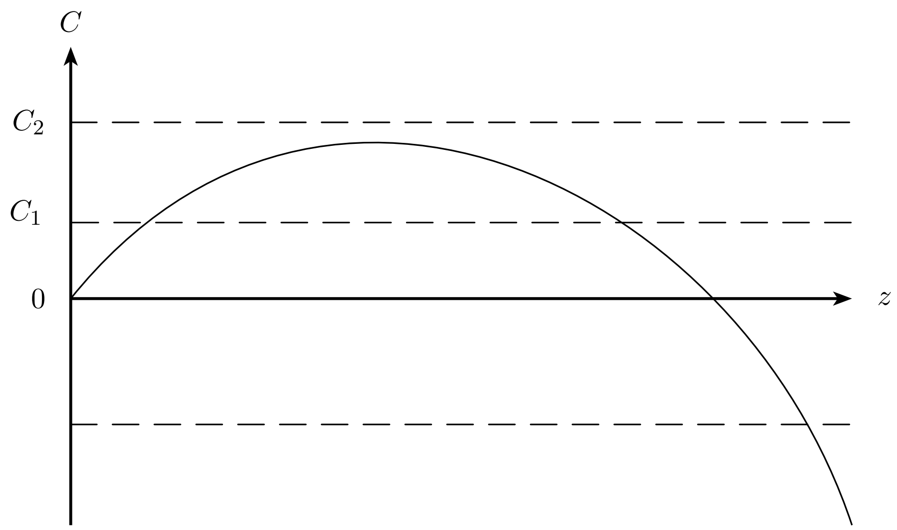

Figure 2.1: Transitions between the corner and interior solutions

As the exponent on equation 2.8's first term is smaller than that on the second, it dominates for small values of z. For larger z, though, the second term overpowers it. Figure 2.1, a stylised plot of equation 2.8, illustrates the implications of this for solutions. For large values of C (e.g. C_2 in the figure) the condition for the corner solution is always satisfied and $x(z) = 0$ is a solution to Bellman's equation. For smaller values, e.g. C_1, it is satisfied for small z, is then violated, and finally is again satisfied for large values of z. For all $C \leq 0$ the condition is violated at small z but eventually comes to hold. It is expected that, when the corner solutions violate condition 2.8, the strategy will continue in the interior.

Interior solutions

The quadratic interior solution, equation 2.7 when $W'(z) \geq -2\xi$, is solved by differentiating it again.[8] The next lemma demonstrates when this is legitimate.

Lemma 2.4 *When $W(z)$ is defined by equation 2.7 and $W'(z) \geq -2\xi$, $W(z) \in \mathcal{C}^\infty$ if*

$$\frac{3}{2}W'(z) + 2\xi - \beta z \neq 0. \tag{2.9}$$

The proof first demonstrates that $W(z) \in \mathcal{C}^2$ when condition 2.9 holds; it then extends this result to $W(z) \in \mathcal{C}^\infty$.

Proof. Define a function, f, such that $W = f(z, W')$ and note that $f \in \mathcal{C}^1$. At points (z_0, W_0') where $f_2 \neq 0$ there exists, by the inverse function theorem, a $g(z_0, W_0) \in \mathcal{C}^1$ such that $W' = g(z, W)$ in the neighbourhood of those points. As g and its arguments are members of $\mathcal{C}^1$ then so is W'; hence $W \in \mathcal{C}^2$ in these neighbourhoods.

Derive an expression for $f_2 \equiv \frac{\partial W(z)}{\partial W'(z)}$ by differentiating equation 2.7 when $W'(z) \geq -2\xi$ with respect to $W'(z)$. This yields

$$\frac{\partial W(z)}{\partial W'(z)} = \frac{1}{\delta}\left[\frac{3}{2}W'(z) + 2\xi - \beta z\right];$$

so that $f_2 \neq 0 \Leftrightarrow$ inequality 2.9.

The result follows by noting that $f \in \mathcal{C}^\infty$. ■

Call the locus of points failing to satisfy inequality 2.9 the *non-invertible locus*.[9] The quadratic term in the interior component of equation 2.7 causes this to pass through the feasible state-action space.[10] On the other hand, when condition 2.9 is not violated, the relevant portion of equation 2.7

[8] This is the approach taken by Tsutsui and Mino [TM90]. Dockner and Sorger [DS96] present a case in which direct integration is possible.

[9] I am grateful to Bernhard von Stengel for suggesting that this be called the *vertible* locus.

[10] In Appendix B the utility function $u(x_1, z) = \ln x_1 - (z - \zeta)^2$ is considered; the non-invertible locus there remains outside the feasible space.

may be differentiated. For notational convenience define $w(z) \equiv W'(z)$. Therefore:

$$w'(z) = \frac{(\beta+\delta)\,w(z) + 2\nu(z-\zeta)}{\frac{3}{2}w(z) - \beta z + 2\xi} \text{ when } x \geq 0. \tag{2.10}$$

Note that the denominator cannot equal zero as that would require $f_2 = 0$ which, by Lemma 2.4, would have prevented the differentiation performed to reach equation 2.10.

To solve equation 2.10 transform the equation into one that is homogeneous of degree zero in its variables by defining $\Omega \equiv w - a$ and $\Psi \equiv z - b$ to drop out its constant terms. This requires that

$$a \equiv 2\nu \frac{\beta\zeta - 2\xi}{\beta(\beta+\delta) + 3\nu}; \tag{2.11}$$

$$b \equiv \frac{2\xi(\beta+\delta) + 3\nu\zeta}{\beta(\beta+\delta) + 3\nu} > 0; \tag{2.12}$$

so that

$$a \leq 0 \Leftrightarrow \text{A1}.$$

These definitions reduce the differential equation to

$$\frac{d\Omega}{d\Psi} = \frac{(\beta+\delta)\,\Omega + 2\nu\Psi}{\frac{3}{2}\Omega - \beta\Psi};$$

or

$$\frac{d\Omega}{d\Psi} = G\left(\frac{\Omega}{\Psi}\right) \equiv \frac{(\beta+\delta)\frac{\Omega}{\Psi} + 2\nu}{\frac{3}{2}\frac{\Omega}{\Psi} - \beta}. \tag{2.13}$$

To take advantage of the homogeneity of equation 2.13 define $S \equiv \frac{\Omega}{\Psi}$. Therefore

$$\left[S^2 - \frac{2}{3}(2\beta+\delta)\,S - \frac{4}{3}\nu\right] d\Psi = \left(\frac{2}{3}\beta - S\right)\Psi dS;$$

which has a trivial solution when $0 = 0$. The RHS zero is attained when S is a constant; that on the LHS is obtained by

$$S = \{s_a, s_b\} \equiv \frac{1}{3}\left[2\beta + \delta \pm \sqrt{(2\beta+\delta)^2 + 12\nu}\right]; \tag{2.14}$$

the real roots of the quadratic in S, with $s_a > 0 > s_b$. The trivial solutions so found are

$$\frac{\Omega}{\Psi} = \{s_a, s_b\};$$

which may be transformed into the original variables for

$$x^a \equiv \xi + \frac{1}{2}\left[a + s_a(z-b)\right]; \tag{2.15}$$

$$x^b \equiv \xi + \frac{1}{2}\left[a + s_b(z-b)\right]. \tag{2.16}$$

Otherwise, when $S \notin \{s_a, s_b\}$, solve

$$\begin{aligned} \frac{d\Psi}{\Psi} &= \frac{\left(\frac{2}{3}\beta - S\right) dS}{\left(S - s_a\right)\left(S - s_b\right)} \\ &= \frac{\gamma_1 dS}{S - s_a} + \frac{\gamma_2 dS}{S - s_b}; \end{aligned} \tag{2.17}$$

when γ_1 and γ_2 are determined by the method of partial fractions to be

$$\begin{aligned} \gamma_1 &\equiv \frac{\delta}{3\left(s_b - s_a\right)} - \frac{1}{2} < 0; \\ \gamma_2 &\equiv \frac{-\delta}{3\left(s_b - s_a\right)} - \frac{1}{2} < 0; \end{aligned}$$

so that $\gamma_1 + \gamma_2 = -1$. Integrating equation 2.17 when $S \notin \{s_a, s_b\}$ then yields

$$\ln|\Psi| = \hat{K} + \gamma_1 \ln|S - s_a| + \gamma_2 \ln|S - s_b|; \tag{2.18}$$

where $\hat{K}$ is a real constant of integration. Exponentiation produces

$$|\Psi| = \frac{1}{K}|S - s_a|^{\gamma_1}|S - s_b|^{\gamma_2}; \tag{2.19}$$

where $K \equiv e^{-\hat{K}} \geq 0$. In terms of z and $W'(z)$ this becomes

$$K = \left|W'(z) - a - s_a(z - b)\right|^{\gamma_1}\left|W'(z) - a - s_b(z - b)\right|^{\gamma_2}.$$

The x^a and x^b solutions correspond to $K = 0$, possible when one of the RHS terms is equal to zero.

Solutions to equation 2.7

To sum up these last two subsections, the solution to differential equation 2.7 is therefore

$$K = |W'(z) - a - s_a(z - b)|^{\gamma_1}|W'(z) - a - s_b(z - b)|^{\gamma_2} \quad \text{when } W'(z) \geq -2\xi; \tag{2.20}$$

and

$$W(z) = -\tfrac{\xi^2 + \nu\zeta^2}{\delta} - \tfrac{\nu}{2\beta + \delta}z^2 + \tfrac{2\nu\zeta}{\beta + \delta}z + Cz^{-\frac{\delta}{\beta}} \quad \text{when } W'(z) \leq -2\xi. \tag{2.21}$$

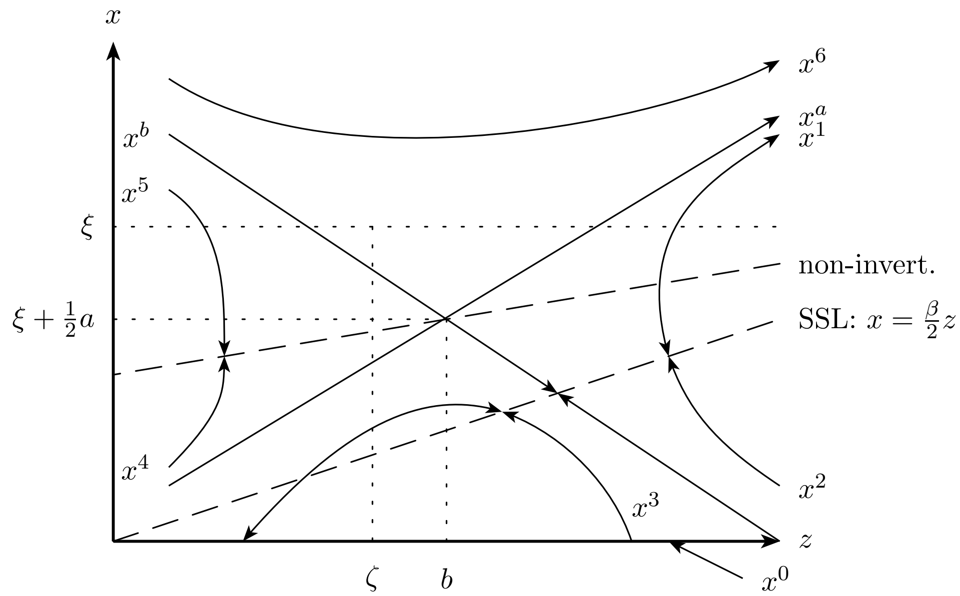

Figure 2.2: Phase diagram generated by differential equation 2.7 when **A1** holds

2.2.4 Defining candidate solution strategies

As K is arbitrary, equation 2.20 describes a family of infinitely many solutions, $\mathcal{X}$, with members x_1. The upward and downward sloping solutions corresponding to $K = 0$ have already been identified as x^a and x^b, respectively. These intersect at $(z, x_1) = \left(b, \xi + \frac{1}{2}a\right) > \mathbf{0}$, inside the feasible (z, x_1) space. Further, when **A1** holds, their intersection is above the climate glut point ($b > \zeta$) and below the product glut point ($\xi + \frac{1}{2}a < \xi$). As there is a non-unique solution to the differential equation at this intersection, call that point a *singularity* and these strategies *singular solutions*. This notation is expanded upon in Chapter 3.

Denote the $x_1(z) = 0$ corner strategy of equation 2.21 by x^0. The remaining six types, denoted $x^1, \ldots, x^6$, are not unique; Figure 2.2 displays representatives of these families. It also displays the steady state locus (SSL), defined by $\frac{dx_1}{dz} = 0$, and the non-invertibility locus, along which $\frac{dx_1}{dz} = \pm\infty$. In the present case, these are $x_1 = \frac{\beta}{2}z$ and $x_1 = \frac{\beta}{3}z + \frac{\xi}{3}$, respectively. Both of these objects appear throughout this chapter; the non-invertibility locus also plays an important role in Chapter 3.

By equation 2.6, the initial Bellman equation's first order condition, $W'(z) < 0$ when $x_1 < \xi$, implying that increases in the initial stock, z, always reduce the value of the game when agent 1 plays at less than the product glut level. This may seem particularly surprising when $z < \zeta$ and above the SSL as z increases in time towards the climate glut point. This benefit is apparently balanced by a loss in the product term and, in some cases, a moving more quickly in time beyond the climate glut point.

A family of indifference curves could also be overlaid on this diagram. They would centre on the glut point (ζ, ξ) and could be parameterised by c:

$$x_1^2 + \nu z^2 - 2\xi x_1 - 2\nu\zeta z + c. \tag{2.22}$$

When $\nu = 1$ these describe circles; otherwise they are ellipses. When $\nu < 1$ deviations from the optimal climate are less costly than those from the optimal output and the ellipses have

$$z = \zeta - \sqrt{\frac{\xi^2 + \zeta^2\nu + c}{\nu(1-\nu)}};$$

as their (vertical) directrix and $\left(\zeta - \sqrt{\frac{\xi^2+\zeta^2\nu+c}{\nu(1-\nu)}}, \xi\right)$ as their foci. When $\nu > 1$ the directrix will be horizontal and the major axis vertical.

Candidate strategies must be able to map from any element of the state space, Z. The x^6 family of strategies, x^0 and x^a (when it does not intersect the horizontal axis) already do so. The interior solutions x^b, x^3 and x^a (when it does intersect the horiztonal axis) are extended by x^0 when they trigger the auxiliary condition, $W'(z) = -2\xi$; denote these extensions by a caret so that $\hat{x}^p \equiv \max\{0, x^p\}$, where p indexes solution families. As these are all integrable, they all give rise to unique solutions to equation of motion 2.3 and therefore represent admissible strategy pairs.

Lemma 2.5 *Members of the x^1, x^2, x^4 and x^5 families of solutions to differential equation 2.7 cannot form candidate MPE strategies.*

Proof. Members of the x^1, x^2, x^4 and x^5 solution families cannot be extended by x^0 as, when they cease to be functions in Z, they do not satisfy the auxiliary condition on $W'(z)$. As they do not specify play for all $z \in Z$ they cannot be candidate strategies. ∎

It is tempting to consider jumps from one of these solutions to, say, x^0. However, no strategy constructed with jumps like this solves differential equation 2.7.

Note also that a candidate MPE strategy cannot switch from x^a to x^b (or vice versa) at their intersection as the ensuing kink would violate Section 2.2.3's requirement that $V(z) \in \mathcal{C}^2$ throughout interior play. Kinks during the transition from interior to corner play are acceptable as **C1** only requires that $V(z)$ is piecewise $\mathcal{C}^1$.

2.2.5 Refining the candidate strategy set

A solution to differential equation 2.7, $W(z)$, is still two steps removed from describing payoffs under MPE play. First, it must be demonstrated that $W(z) = V(z)$, that the candidate value function is a value function. Lemma A.1 in Appendix A provides necessary and sufficient conditions for this identification. Second, play arising from the value functions must be best responses to each other, not merely to some fixed play. Theorem A.2, also in Appendix A, provides a sufficient condition for this.

Most of the candidate MPE strategies are discarded here without direct reference to the appendix' results. Those results are more useful when attention turns from discarding possible strategies to proving that $\hat{x}^b$ and, occasionally, some of the $\hat{x}^3$ candidate do support MPE.

Glut point and corner deviations: $\hat{x}^a, \hat{x}^6, x^0$ and $\hat{x}^3$

Lemma 2.6 *$W(z) \neq V(z)$ along $\hat{x}^a$ and the x^6 family of strategies.*

Proof. Along $\hat{x}^a$: $z \to \infty \Rightarrow W'(z) \to \infty \Rightarrow W(z) \to \infty$, an impossible integral of the bounded above instantaneous utility function 2.1. As $x^6(z) > \hat{x}^a(z)$, the x^6 family produces the same contradiction. ■

As the lemma and theorem of Appendix A provide conditions whereby candidate strategies may be dismissed from further consideration, it might be thought that Lemma 2.6 interacts with the results in the appendix. It does not: Lemma A.1 is inapplicable to this situation due to the indeterminate $\lim_{T\to\infty} W(z(T))$ along these strategies. For the same reason, Theorem A.2 would not assist even were $W(z)$ a value function.

The argument that $\hat{x}^a$ and the $\hat{x}^6$ family do not provide candidate MPE strategies may be illustrated by demonstrating a profitable deviation from their play: as $s_a > \beta$, there is a z such that $\dot{z} > 0$ and $x(z) > \xi$ for all greater values of z along these strategies. An agent can then improve its payoff by capping play at $x_i = \xi$; doing so sets the utility loss term in production to zero and slows the climate loss term's growth (as compared to playing $x_i > \xi$).

Discarding $\hat{x}^a$ and the x^6 family leaves only the x^0, $\hat{x}^b$ and the $\hat{x}^3$ family of strategies to consider as possible MPE strategies.

Lemma 2.7 *$W(z) \neq V(z)$ in any candidate that satisfies $x(0) = 0$ and possesses constant of integration $C \neq 0$ in that cornered component.*

Proof. By equation 2.21

$$\lim_{z\to 0} W(z) = -\frac{\xi + \nu\zeta^2}{\delta} + C \lim_{z\to 0} \frac{1}{z^{\frac{\delta}{\beta}}}.$$

When $C > 0$, this unbounded limit again contradicts the bounded above instantaneous utility function. As noted in equation 2.8, which provided the condition for the solution to equation differential 2.7 to remain in the corner, $C < 0$ and $x(0) = 0$ are contradictory. ■

The candidate with $C = 0$ is not eliminated by Lemma 2.7; it leaves $x(0) = 0$ immediately.

Again, express this rejection of $x(0) = 0$ in terms of profitable deviations by considering play at $z < \zeta$, the glut climate. The cornered strategy requires that agent 1 accept a climate loss as z continues to fall; defection to some small $x_1 > 0$ reduces the climate loss and provides a production gain.[11]

Therefore:

Lemma 2.8 $\hat{x}^b$ *is the only remaining candidate strategy when*

$$a - s_a b \leq -2\xi. \tag{2.23}$$

Proof. As the $\hat{x}^3$ strategies are bounded above by $\hat{x}^a$, Lemma 2.7 rules out all $\hat{x}^3$ when $\hat{x}^a(z) = 0$ for $z > 0$; this happens when condition 2.23 holds. ■

Condition 2.23 is used later in this chapter to explore conditions under which the game yields a unique MPE.

MPE strategies: $\hat{x}^b$ and $\hat{x}^3(0) > 0$

Having discarded various families of solutions from further consideration, this subsection now establishes positive results, proving that certain candidates do support MPE.

Theorem 2.9 $\hat{x}^b$ *and any* $\hat{x}^3(0) > 0$ *candidates are MPE strategies.*

The results in Appendix A are necessary to prove this result. Lemma A.1 is satisfied by these strategies: they are continuous and cause the state variable to converge to a finite z^*. Theorem A.2 requires a finite limit on the value function against deviant strategies, and therefore any $z \in Z$. As the value functions under consideration here are continuous, it is sufficient to test them at extreme values of z, zero and infinity. As the instantaneous utility function is unbounded below, the limit is not obviously satisfied if the deviant strategy being considered sets $z_T = \infty$.

To circumvent this problem, a modified game is introduced here. Two lemmata then prove that results on the modified game apply to the original game.

[11] Similar reasoning would also apply to an $\hat{x}^3$ member for which $x(0) > 0$ but which then declined to $x(z) = 0$ at some $0 < z < \zeta$. Equation 2.8 reveals that this is an impossibility: the $\hat{x}^3$ path that passes through $(0, 0)$, and therefore attains $x(z) = 0$ at the lowest z, is identified by $C = 0$ along its corner component. This constant sets $x(z) = 0$ at $z \in \left\{0, \frac{2\beta+\delta}{\nu}\left(\xi + \frac{\nu\zeta}{\beta+\delta}\right)\right\}$. As this second value exceeds ζ for non-negative parameters, the impossibility is established.

Therefore consider a bounded version of the instantaneous utility function 2.1,

$$\underline{u}(x_1, z) = \max\{\underline{u}, u(x_1, z)\};$$

where the finite constant $\underline{u}$ is chosen so that Bellman's equation produces the same set of candidate strategies as under the original instantaneous utility function; this merely requires that $\underline{u}$ be sufficiently negative to ensure that it is never attained by candidate play. It may need to be very large and negative if the initial stock level is very large.

Lemma 2.10 *$\hat{x}^b$ and the any $\hat{x}^3(0) > 0$ members from the original model represent MPE play in the game with the modified instantaneous utility function.*

Proof. As the candidate set, $\underline{\mathcal{W}}$, is the same as in the unmodified game, the same refinements may be performed, leaving $\hat{x}^b$ and the remaining $\hat{x}^3(0) > 0$ members from the original model as candidates. As the modified instantaneous utility function is bounded, the value functions of these strategies, $\underline{V}(z)$, are also bounded. Any play therefore generates a bounded value, satisfying Theorem A.2. ■

Lemma 2.11 *MPE strategies in the modified model are MPE strategies in the original game.*

Proof. By choice of $\underline{u}$, the payoffs of candidates strategies are the same in each model so that $V(z) = \underline{V}(z)$ for any given candidate strategy. Further, as the instantaneous payoff to any strategy under the bounded model is no worse than that under the original, $V(z) \leq \underline{V}(z)$ for all strategies, including deviations. If deviations are not sufficient to eliminate the $\hat{x}^b$ and surviving $\hat{x}^3$ strategies in the modified model and are no more profitable in the original model then the $\hat{x}^b$ and surviving $\hat{x}^3$ strategies survive in it as well and are therefore MPE strategies. ■

Therefore $\hat{x}^b$, the downward-sloping linear solution, is not just a MPE when strategies are restricted to be linear (a well known result; q.v. [BO99, p. 324]. The sufficient conditions presented by Lockwood [Loc96] for a unique affine equilibrium strategy are satisfied here.), but is also in the larger space of piecewise $\mathcal{C}^1$ strategies. As condition 2.23, when it holds, leaves $\hat{x}^b$ as the unique MPE strategies, extreme cases are now explored to see when it might be the only solution.

2.2.6 Unique MPE strategies?

In general, both linear and non-linear strategies may support MPE in the linear-quadratic game. This section explores certain hypothetical extreme cases in an attempt to develop an intuition for when the linear alone survives. They are hypothetical as they set a parameter to zero, in contravention of the previous requirement that these all be strictly positive.

Perfectly patient agents ($\delta = 0$)

First consider perfectly patient agents, $\delta = 0$. Substitution into condition 2.23 now produces the quadratic in ξ

$$\xi^2 \left(\beta^2 - \nu\right) + \xi \left(4\beta\nu\zeta\right) + 3\nu^2\zeta^2 \geq 0. \tag{2.24}$$

This holds when

$$\frac{\zeta}{\xi} \geq \frac{\sqrt{\beta^2 + 3\nu} - 2\beta}{3\nu};$$

a sufficient condition for which is that $\beta^2 \geq \nu$. In these cases $\hat{x}^b$, the strategy with the greatest steady state stock level, is the unique MPE strategy. An interpretation of this result is that large climate glut (ζ) relative to product glut (ξ) and rapid degradation (β) relative to climate sensitivity (ν) favour high steady state stocks levels. This interpretation is not, however, an explanation.

Can the first best be attained? Perfectly patient agents suggest the possibility that the first best steady state of equation 2.4, (z_{fb}, x_{fb}), might be attained by a MPE strategy; it cannot be reached by unilateral deviation by a single agent as it is symmetric.

Lemma 2.12 *When assumption* ***A1*** *holds with equality, the steady state of the linear MPE, $\hat{x}^b$, is that of the first best for perfectly patient agents.*

Proof. When agents are perfectly patient, the state at the first best identified in equation 2.4 reduces to

$$z_{fb} = 2\frac{\beta\xi + 2\nu\zeta}{\beta^2 + 4\nu}.$$

Similarly, substitution of the expressions for a, b and s_b in Section 2.2.3, finds the intersection of $\hat{x}^b$ and the SSL for perfectly patient agents to be

$$z = \frac{6}{\beta^2 + 3\nu} \frac{\frac{1}{3}\beta^2\xi + \nu\xi + \left(\frac{2}{3}\beta\xi + \nu\zeta\right)\sqrt{\beta^2 + 3\nu}}{\beta + 2\sqrt{\beta^2 + 3\nu}}.$$

When $\beta\zeta = 2\xi$ is imposed with equality, these reduce to

$$\begin{aligned} z_{fb} &= \frac{\frac{\xi^2}{\zeta} + \nu\zeta}{\frac{\xi^2}{\zeta^2} + \nu} = \frac{1}{\frac{1}{\zeta}} = \zeta; \text{ and} \\ z &= \frac{4\frac{\xi^3}{\zeta^2} + 3\nu\xi + \left(4\frac{\xi^2}{\zeta} + 3\nu\zeta\right)\sqrt{4\left(\frac{\xi}{\zeta}\right)^2 + 3\nu}}{4\left(\frac{\xi^3}{\zeta^2}\right)\frac{1}{\zeta} + 3\nu\xi\frac{1}{\zeta} + \left(4\left(\frac{\xi^2}{\zeta}\right)\frac{1}{\zeta} + 3\nu\zeta\frac{1}{\zeta}\right)\sqrt{4\left(\frac{\xi}{\zeta}\right)^2 + 3\nu}} = \zeta; \end{aligned}$$

respectively. ■

Similar calculations reveal that, when **A1** holds with strict inequality, (z_{fb}, x_{fb}) has a lower stock and emissions level; the opposite occurs when **A1** does not hold. In the former case, there may be $\hat{x}^3$ MPE strategies whose steady states coincide with the first best; in the latter, there are certainly not. Therefore, only when **A1** holds with strict inequality is it possible for the steady state of an $\hat{x}^3$ MPE to be preferred to that of $\hat{x}^b$.

Unilateral global deviations from the steady state A perfectly patient agent may also contemplate global deviations from the steady state. Define a new locus, the asymmetric steady state (ASSL), such that

$$x_A + \hat{x} - \beta z = 0; \tag{2.25}$$

where $\hat{x}$ is the strategy played by the non-deviating agent and x_A is that played by the deviant. Given some $\hat{x}$, then, this locus is unique; it intersects the SSL defined by $x = \frac{\beta}{2}z$ at $x_A(z) = \hat{x}(z)$. The ASSL is the locus of steady states attainable by unilateral deviation. A perfectly patient agent may therefore ask whether it contains points that yield a higher steady state payoff than that resulting from play of $\hat{x}$.

To determine this, allow agent 1 to maximise its instantaneous utility, subject to the constraint that its choice lie on the ASSL. The agent's problem is therefore to

$$\max_z \left[-\left(\beta z - \hat{x}(z) - \xi\right)^2 - \nu (z-\zeta)^2\right];$$

which yields first order conditions

$$(\beta z - \hat{x} - \xi)\left(\beta - \frac{d\hat{x}}{dz}\right) + \nu(z-\zeta) = 0.$$

The relationships in equations 2.6, the first order conditions from the original Bellman equation, and 2.10, the differential equation describing interior play, allow this to be rewritten as

$$(\beta z - \hat{x} - \xi)\left(\beta - \frac{\beta(\hat{x}-\xi) + \nu(z-\zeta)}{3\hat{x} - \beta z - \xi}\right) + \nu(z-\zeta) = 0;$$

which simplifies to

$$\left[\beta(\beta z - x - \xi) + 2\nu(z-\zeta)\right]\left[2x - \beta z\right] = 0. \tag{2.26}$$

Therefore, $x = \frac{\beta}{2}z$, the SSL, satisfies this. As the original constraint required the choice to lie along the ASSL as well, one solution to agent 1's optimisation problem is that dictated by non-deviating play of $\hat{x}$. Another solution to equation 2.26 is

$$x = \beta z - \left(\xi - \frac{2\nu(z-\zeta)}{\beta}\right).$$

As the question of when this line intersects with a candidate $\hat{x}$ becomes mired in algebra, it is not pursued.

As a final note, a linear candidate strategy is sufficient for the solutions to equation 2.26 to be maxima. This is readily seen by noting that the second order conditions require

$$\left(\beta - \hat{x}'\right)^2 - \hat{x}'' \left(\beta z - \hat{x} - \xi\right) + \nu \geq 0.$$

No environmental decay ($\beta = 0$)

Substitution into condition 2.23 and manipulation now yields a condition in the relative glut levels

$$\frac{\zeta}{\xi} \geq \frac{1}{6\nu} \left[\sqrt{\delta^2 + 12\nu} - 5\delta\right]. \tag{2.27}$$

Preferences for a greenhouse gas saturated atmosphere, without the possibility of reduction, make it more likely that only $\hat{x}^b$ survives. A sufficient condition for only $\hat{x}^b$ to survive is that the right hand side of inequality 2.27 be negative. This requires that $2\delta^2 \geq \nu$: high impatience relative to environmental sensitivity only leaves the candidate with the highest steady state stock level, $\hat{x}^b$. Again, this interpretation does not explain why only the linear MPE strategy survives.

Climate insensitive ($\nu = 0$)

Now inequality 2.23 reduces to the condition that

$$\beta + 2\delta \geq 0;$$

which holds for all non-negative parameters. In these cases $\hat{x}^b$, the unique MPE strategy, reduces to $x(z) = \xi$: without a stock effect, there is no interaction between the agents; they simply optimise with respect to production.[12]

Equilibria when assumption A1 fails

Assumption **A1** on page 21 required that $\beta\zeta \leq 2\xi$. Until now, it has been assumed to hold. As values of β, ξ and ζ can cause it not to, a number of consequences of its failure are now examined to see whether this alters any of the results above. Consequences of **A1**'s failure include:

1. $a > 0$ (equation 2.11);
2. x^a and x^b intersect at a higher stock and emissions level than the glut levels;

[12] This result replicates that of Dockner and Long [DL93].

3. x^a and x^b intersect below the SSL.

As the slope of the SSL is less than that of x^a for all non-negative parameters, x^a therefore intersects the horizontal axis, satisfying condition 2.23, that excluding the $\hat{x}^3$ strategies. As the techniques used to discard candidate strategies when **A1** held rule out the same strategies when **A1** is violated, $\hat{x}^b$ is the unique MPE.

2.2.7 Does the world have a unique equilibrium?

Clearly, the model developed here is incredibly crude when compared to the interactions around GHG that may be taking place in the world today. Nevertheless, an attempt at calibration is made here to see whether the continuum of MPE might be expected to survive under plausible parameter values.

The longest atmospheric carbon dataset is the Vostok ice core set, which extends back some 420,000 years.[13] It reveals the historical range of atmospheric carbon to be about $(380, 760)$ Gt, the upper limit only now being reached. The series appears stationary over this period. While a half million years is not a long period geologically, and it is not clear whether the series is stationary over longer periods, this range seems as good a guess as any for the optimal greenhouse gas stock, ζ. Therefore assume $\zeta \in [380, 760]$.

Calibrating ξ is equally questionable as ξ represents the optimal production of fictitious symmetric agents. Here it is more reasonable to assume that optimal production is close to what nations actually do but the question becomes one of what they actually do. One option is to use the USA, the world's largest emitter, as one agent and the rest of the world as the other. The USA emitted roughly 1.4 Gt of carbon in 1990 while the world emitted 6.1 Gt of carbon from industrial processes' carbon dioxide in 1991.[14] A range for ξ might therefore be $\xi \in [1.4, 3.0]$, the lower bound coming from the US emissions and the upper from half the global total (the world being divided into two symmetric agents).

The relative concern for industrial output and environment amenities, ν, require similarly heroic assumptions. The estimate here is based on the consensus finding of economic studies that a doubling of pre-industrial CO_2 levels (600 Gt) will lead to a 1 - 2% GDP loss [Int95]. If a doubling of the stock from 600 Gt has the same effect on instantaneous utility as a 1 - 2%

[13] Atmospheric carbon is not the only GHG but its product, CO_2, is commonly used as a proxy for the others; that practice is followed here. While less abundant than atmospheric carbon, methane and CFCs are, per molecule, more potent in terms of their contribution to the greenhouse effect than CO_2. A stock estimate based on Vostok data therefore likely understates 'effective' stock.

[14] The source for the data used here is the UNEP webpage, `http://www.unep.ch/iucc/fact30.html`, unless otherwise noted.

production decrease then

$$-\left(L\hat{x}-\xi\right)^2-\nu\left(z_0-\zeta\right)^2=-\left(\hat{x}-\xi\right)^2-\nu\left(2z_0-\zeta\right)^2;$$

where $z_0 = 600$ Gt, the preindustrial stock, $\hat{x}$ is current production (if it is optimal when the atmosphere is ignored $\hat{x} = \xi$) and L is the production loss coefficient, $L \in \left\{\frac{98}{100}, \frac{99}{100}\right\}$. This may be solved for

$$\nu=\frac{(L-1)\,\hat{x}}{z_0}\frac{(L+1)\,\hat{x}-2\xi}{3z_0-2\zeta}.$$

If $\hat{x} = \xi$, $z_0 = 600$, ζ and L range as above and $\nu \geq 0$ then $\nu \in \left(0, 2.1 \times 10^{-8}\right]$.

Nordhaus' work [Nor94] is vandalised to estimate β. His discrete equation of motion for greenhouse gas stocks is

$$M\,(t)-590=\beta\cdot E\,(t-1)+(1-\delta_M)\cdot[M\,(t-1)-590];$$

where $M\,(t)$ are atmospheric stocks, β the atmospheric uptake ratio, $E\,(t-1)$ the previous period's emissions and δ_M the rate of transfer from the upper reservoir (the atmosphere) to the lower reservoir (the oceans and soils). He estimates $\delta_M = .0083$ per decade (the units in which he measures time). Ignoring his 590 threshold, assume that the fraction of the stock retained by the atmosphere after a decade is $(1 - .0083)$ in the absence of further emissions. In the present notation again

$$\begin{aligned} z\,(t) &= e^{-\beta t}z_0 \\ z\,(10) &= .9167z_0 = e^{-10\beta}z_0; \end{aligned}$$

or $\beta \approx .0087 \approx \frac{1}{115}$ (as t is measured in years).

A range of plausible values for δ may be derived from the Euler equation of the life-cycle savings problem in the utility function $u\,(c)$, where c is consumption, $u' > 0$ and $u'' < 0$:

$$\delta=r+\frac{u''\,(c)}{u'\,(c)}g\cdot x^0\cdot e^{gt};$$

where r is the real rate of interest facing consumers and g the real growth rate. Therefore $\delta \leq r$ and could be as low as zero. Consider $\delta \in (0, .02]$, where the upper bound is an estimate of the real rate of interest.

Now test condition 2.23 by rearranging it for

$$a-s_ab+2\xi\leq 0;$$

and maximising the left hand side by choice of δ, ν, ξ and ζ, as constrained above. As a point estimate exists for β, it is not regarded as a choice variable. This maximisation is a harsher test than necessary as ν depends on ξ and ζ; this harshness is repaid, though, by allowing the problem to be treated as one that is linear in ξ and ζ. Maximisation may therefore take place with respect to these independently of choice of δ and ν. The extremal values $\xi^* = 3$ and $\zeta^* = 380$ are found.[15]

Now maximise by choice of δ and ν by writing the Lagrangian

$$\begin{aligned}\mathcal{L}(\delta, \nu, \lambda, \gamma) = & 3\left\{2\nu + \frac{2}{3}\left(\frac{1}{115} + \delta\right)\left(\frac{1}{115} - \delta - \sqrt{\left(\frac{2}{115} + \delta\right)^2 + 12\nu}\right)\right\} \\ & -380\nu\left(\delta + \sqrt{\left(\frac{2}{115} + \delta\right)^2 + 12\nu}\right) \\ & -\lambda_1\left[\delta - \frac{1}{50}\right] + \lambda_2\delta - \gamma_1\left[\nu - 5.4 \times 10^{-7}\right] + \gamma_2\nu.\end{aligned}$$

There are nine cases to consider, with each of δ and ν taking on an extremal value at their upper or lower bounds or in the interior of their ranges. An interior ν when $\delta = 0$ turns out to require a complex ν, a contradiction. Both other interior cases for ν produce inadmissible signs on multipliers, also contradictions. When $\nu = 0$, only $\delta = 0$ does not violate similar sign conditions. Finally, when $\nu = 5.4 \times 10^{-7}$, all cases of δ yield sign contradictions. Therefore $(\delta^*, \nu^*) = \mathbf{0}$ solves this problem and the harshest combination of variables to which condition 2.23 can be subjected is $(\beta, \delta, \nu, \xi, \zeta) = \left(\frac{1}{115}, 0, 0, 3, 380\right)$. As this sets

$$a - s_a b + 2\xi - \frac{2}{3}\xi^*(1 + 4\beta^*) \approx -2 < 0;$$

condition 2.23 always holds.

This result allows a conclusion to be drawn from this thought experiment. If it is believed that the linear-quadratic model presented here might bear some resemblance to the real world, then the calibration attempted in this section suggests that the real world is likely to satisfy the conditions for a unique MPE, that in linear strategies.

2.3 A nonlinear model: finite atmospheric capacity

The model examined to date gave the atmosphere the capacity to absorb an infinite greenhouse gas stock. This assumption allowed use of a linear-quadratic game but is unrealistic in the present context for at least two reasons. First, the world only contains a finite supply of carbon and the other constituents of greenhouse gases. Second, the atmosphere will become saturated, shedding stock more quickly, as concentrations climb. This

[15]When the coefficient of ξ is negative, condition 2.23 is automatically satisfied.

second observation suggests a change in specification of the decay term of the equation of motion, $\beta z(t)$. This section therefore illustrates the consequences of a non-constant decay term. As the candidate strategies derived are more complicated than those considered to date, no attempt is made to present the MPE set.

Consider $\beta(z) \equiv \frac{\theta z(t)}{\bar{z}-z}$, where $\theta > 0$. Therefore $\beta'(z) > 0$, reaching a vertical asymptote at $\bar{z}$, atmospheric capacity. The new state space is $\bar{Z} \equiv (0, \bar{z})$. The new equation of motion,

$$\dot{z} = x_1(t) + x_2(t) - \frac{\theta z(t)}{\bar{z} - z(t)} \text{ s.t. } z(0) = z > 0; \tag{2.28}$$

removes the game from the linear-quadratic world but does not change the first order necessary conditions (equation 2.6) of the new Bellman equation. Note that $x_1 = x_2 = z = 0 \Rightarrow \dot{z} = 0$. As equation 2.28 cannot be solved as easily as was its predecessor, equation 2.3, admissibility of strategies is also more complicated than it was in Section 2.2. Cauchy's Theorem for a unique solution requires that $x_i \in \mathcal{C}^1, i \in 1, 2$ and $z(t) \neq \bar{z}$; these are taken as necessary conditions for admissible strategies.

To proceed, again invoke symmetry and substitute in the first order conditions.

The corner solution still holds when $W'(z) \leq -2\xi$; it produces the differential equation

$$\delta W(z) = -\xi^2 - \nu(z-\zeta)^2 - W'(z)\frac{\theta z}{\bar{z}-z};$$

from the Bellman equation. The solution to this is

$$W(z) = \frac{1}{\theta}\left(\frac{e^z}{z^{\bar{z}}}\right)^{\frac{\delta}{\theta}}\left\{C\theta + \int \frac{(\bar{z}-z)}{z}\left(\frac{z^{\bar{z}}}{e^z}\right)^{\frac{\delta}{\theta}}\left[-\xi^2 - \nu(z-\zeta)^2\right]dz\right\};$$

where C is a constant of integration. Again, the condition on $W'(z)$ imposes a condition on z for any C.

In the interior portion of the solution the differential equation, differentiate again for

$$w'(z) = \frac{\left(\delta + \frac{\theta}{(\bar{z}-z)^2}\right)w + 2\nu(z-\zeta)}{\frac{3}{2}w + 2\xi - \frac{\theta z}{\bar{z}-z}}. \tag{2.29}$$

This second differentiation is permitted by the assumption that $x_i \in \mathcal{C}^1$. In (z, x_1) space, equation 2.29 is

$$\frac{dx_1}{dz} = \frac{\left((\bar{z}-z)^2\delta + \theta\right)(x_1 - \xi) + \nu(z-\zeta)(\bar{z}-z)^2}{(\bar{z}-z)\left[(3x_1 - \xi)(\bar{z}-z) - \theta z\right]}; \tag{2.30}$$

when $x_1 > 0$.

As analytic solutions to these differential equations are not found, standard phase diagram techniques are used. First, define loci based on setting the numerator and denominator of equation 2.30 to zero. The non-invertibility locus, $D(z)$, sets the denominator to zero:

$$D(z) \equiv \frac{1}{3}\left\{\xi + \frac{\theta z}{\bar{z} - z}\right\};$$

whenever $z \neq \bar{z}$. Therefore $D'(z) > 0 \forall z \in \bar{Z}, D(0) = \frac{1}{3}\xi$ and $D(z) = 0$ is solved uniquely by $z = \frac{\xi}{\xi - \theta}\bar{z} \notin \bar{Z}$.

The locus of points setting the numerator to zero, $N(z)$, is

$$N(z) \equiv \xi - \nu \frac{(z - \zeta)(\bar{z} - z)^2}{(\bar{z} - z)^2 \delta + \theta}.$$

Thus $N(z) \in \mathcal{C}^1 \forall z \in \bar{Z}, N(0) = \xi + \nu \frac{\zeta \bar{z}^2}{\delta \bar{z}^2 + \theta}$ and $N(\zeta) = N(\bar{z}) = \xi$. As $N(\bar{z})$ is a local maximum, a local minimum occurs at some $z_1 \in (\zeta, \bar{z})$.[16] $N(z)$, as a cubic, has up to three real roots.

Finally, the steady state locus (SSL) is $S(z) \equiv \frac{\theta z}{2(\bar{z} - z)}$. Therefore $S(z) \in \mathcal{C}^1, S'(z) > 0 \forall z \in \bar{Z}$ and $S(0) = 0$. Note that $D(z_2) = S(z_2) = \xi$, where $z_2 \equiv \frac{2\xi}{2\xi + \theta}\bar{z} \in \bar{Z}$. As the intersections of $N(z)$ and the other two loci are more complicated, characterisation is not attempted.

The argument for refining the strategies is developed graphically, with reference to Figure 2.3. The main loci, $N(z)$, $D(z)$ and $S(z)$, are displayed; the signs of the former two, which set the numerator and denominator in equation 2.30 to zero, are indicated off the loci. A few strategies are sketched in to provide a sense of the dynamics. As they resemble those in the linear world of Figure 2.2, the same labelling convention is used.

The configuration displayed in Figure 2.3 is not the only one possible. For example, $N(z)$ may intersect the horizontal axis or $N(z)$ and $D(z)$ may intersect each other above $x_1 = \xi$. The analysis presented below, though, includes these cases as well. As above, the focus is on eliminating candidate strategies.

First eliminate strategies intersecting $D(z)$ at any point other than its intersection with $N(z)$: as in the linear world, these are not functions.

As in Section 2.2.5, the x^a and x^6 strategies may also be eliminated. This is done by considering the same deviation as there, capping play at $x_1(z > \zeta) = \xi$. Lemma 2.6 does not apply when the state space is bounded above.

---

[16] The root, z_1 solves the cubic

$$\delta z_1^3 + (3\delta\bar{z}) z_1^2 - 3\left(\delta\bar{z}^2 + \theta\right) z_1 + \delta\bar{z}^3 + (2\zeta + \bar{z})\theta = 0;$$

when $z \neq \bar{z}$.

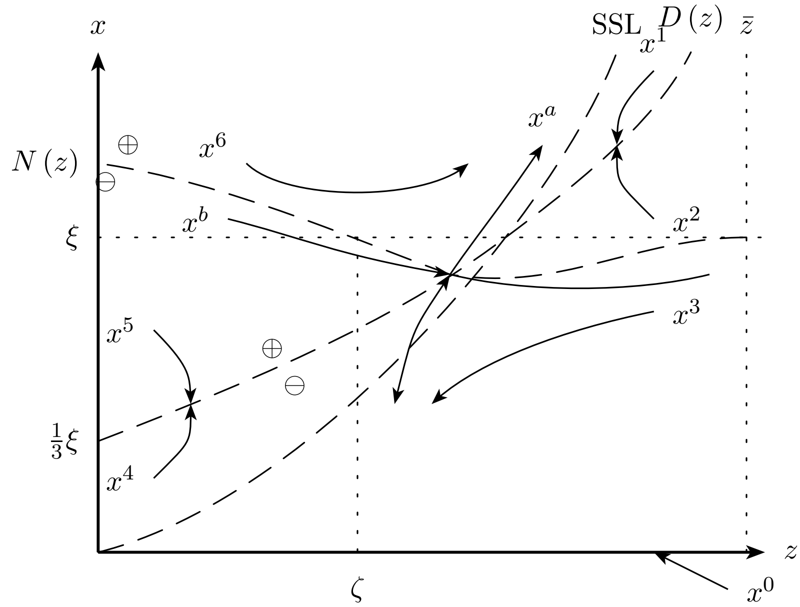

Figure 2.3: A representative world with non-linear equation of motion, 2.28

Finally, eliminate x^0 and all x^3 strategies for which $\hat{x}^3\left(z<\zeta\right)=0$, where the caret indicates an extension to the corner solution. Again, these can be profitably deviated from when $z<\zeta$. As before, all $\hat{x}^3$ strategies are eliminated when $\hat{x}^a$ intersects the horizontal axis within the state space. The condition for which this holds, although complicated, can be calculated from differential equation 2.30.

Therefore the only candidates that may survive are $\hat{x}^b$ and some of the $\hat{x}^3$. Their survival depends on Lemma A.1 and Theorem A.2. As their application requires more information on the candidate strategies' behaviour, no further statements are made about them here.

2.4 Discussion

The key result of this chapter is that, under certain calibrations, the linear-quadratic game has a unique linear MPE while, under the remaining calibrations, it has a continuum of non-linear MPE as well. The latter case may seem peculiar as the linear $\hat{x}^b$ strategy and the non-linear $\hat{x}^3$ strategies appear quite different in (x,z) space. They appear more similar, though, when regarded as being parameterised by K, the constant of integration in equation 2.20. This allows $\hat{x}^b$ to be seen as the limit of the $\hat{x}^3$ strategies as K goes to zero.

The dependence of the equilibrium set on calibration means that the correspondence mapping between the parameters of this game and its MPE set is not continuous, the latter exploding when the uniqueness condition fails. The strict inequality in that condition (condition 2.23 on page 30), meaning that the explosion requires $a - s_a b > -2\xi$, makes the candidate correspondence lower hemi-continuous but not upper hemi-continuous (as the graph is not closed). The game is therefore not *essential* in the sense of Fudenberg and Tirole [FT91, Definition 13.6]: small perturbations to the parameters of the objective function may cause large perturbations to the MPE set.

These results are more complicated than those presented in Tsutsui and Mino [TM90]. While the continuum result does not survive all calibrations of the linear-quadratic model, it does seem robust to a variety of game specifications; that considered in Appendix B yields a continuum of MPE, as does that analysed in Dockner and Sorger [DS96].

The explanation that Tsutsui and Mino gave for the survival of the continuum is straightforward: the transversality conditions applied were insufficient to eliminate all but a unique solution to the differential equation. A similarly technical answer exists to explain the present phenomenon: sometimes the various filters applied are sufficient to eliminate all but a linear solution; otherwise, they are not.

For the most part, no economic intuition has been found for these results. One minor exception to this is the case in which agents are insensitive to climate ($\nu = 0$). In this degenerate case, the game reduces to a static optimisation problem in which each agent minimises its production loss by choice of production. This yields the unique solution which sets production loss to zero.

Before considering a full treatment of asymmetric nations, developed over the next chapters by numerical analysis, a special case is now considered in Appendix C; that asks whether corner play by one agent may be an equilibrium strategy.

Appendix A

Transversality conditions

This appendix presents a lemma and a theorem. The lemma provides a necessary and sufficient condition for a candidate value function, $W()$, to be a value function, $V()$. The theorem provides a sufficient condition for a strategy pair, $(x_1^*(), x_2^*())$, to support a MPE. Both involve limits on (candidate) value functions.[1] In the lemma, value has the sense of 19, the present discounted value of utility. In the theorem, value has the sense of 21, the best response to fixed play by the other agent.

Both results apply to the more general utility function $u(x, z)$; neither requires nor implies symmetry of agents and both hold for any equation of motion that is linear in its controls.

Lemma A.1 *Given two admissible strategies, $\hat{x}_1$ and $\hat{x}_2$, the instantaneous objective function $u(x_i, z)$, where z is a state variable whose equation of motion, $\dot{z}(x_1, x_2, z)$, is linear in the controls x_1 and x_2, if*

$$\delta W_1(z) = u(\hat{x}_1, z) + W_1'(z)\, \dot{z}(\hat{x}_1, \hat{x}_2, z)\,; \tag{A.1}$$

then a necessary and sufficient condition for $W_1()$ to be a value function, $V_1()$, is that $\lim_{T\to\infty} e^{-\delta T} W_1(\hat{z}_T) = 0$, where $\hat{z}_T$ is the state after play of $\hat{x}_1()\ \hat{x}_2()$ over $t \in [0, T]$ from initial state z_0.

Proof. Rewrite equation A.1 by collecting the terms in $W_1(z)$ and multiplying by $e^{-\delta t}$ for

$$\frac{d}{dt}\left[e^{-\delta t} W_1(z)\right] = -e^{-\delta t} u(\hat{x}_1, z)\,.$$

Integrate over $[0, T]$ for

$$W_1(z_0) = \int_0^T e^{-\delta t} u(\hat{x}_1, z)\, dt + e^{-\delta T} W_1(\hat{z}_T)\,.$$

[1] Tsutsui and Mino's existence proof [TM90, Theorem 1] does not explicitly consider limits but that paper's bounded state space plays the same role.

As this becomes

$$W_1(z_0) = \int_0^\infty e^{-\delta t} u(\hat{x}_1, z)\, dt + \lim_{T\to\infty} e^{-\delta T} W_1(\hat{z}_T); \tag{A.2}$$

in the limit and the first term on the right hand side is the payoff to optimal play it is necessary and sufficient that $\lim_{T\to\infty} e^{-\delta T} W_1(\hat{z}_T) = 0$ for $W_1(z)$ to be a value function, $V_1(z)$, the sum of discounted utility. ■

When $\hat{x}_1() = x_1^*()$, defined according to equation 3.3, the $W_1()$ functions defined in equation A.1 are solutions to the differential equation generated by the Bellman equation.

The next theorem asks whether a sustained deviation, x_1, can obtain a superior payoff to an x_1^* satisfying the Bellman equation against x_2^* by effecting a global deviation before returning to optimal play. In other words, is x_1^* a best response to x_2^*?

Theorem A.2 *Given the instantaneous objective function and equation of motion of Lemma A.1, any admissible strategy $x_2^*(z)$, an admissible strategy $x_1^*(z)$ satisfying*

$$\begin{aligned} \delta V_1(z(t)) = &\ u(x_1^*(z(t)), z(t)) + V_1'(z(t))\, \dot{z}(x_1^*, x_2^*, z) \\ &\geq u(x_1(z(t)), z(t)) + V_1'(z(t))\, \dot{z}(x_1, x_2^*, z); \end{aligned} \tag{A.3}$$

for all admissible $x_1(z)$, then a sufficient condition for $x_1^*(z)$ to be a best response to $x_2^*(z)$ is that $\lim_{T\to\infty} e^{-\delta T} V_1(z_T) \geq 0$, where z_T is the result of any play, $x_1(z(t))$ against $x_2^*(z(t))$ over $t \in [0, T]$ from initial state z_0.

Proof. Differentiate the discounted value function for

$$\begin{aligned} \frac{d}{dt}\left[e^{-\delta t} V_1(z(t))\right] &= e^{-\delta t}\left[V_1'(z(t))\, \dot{z}(x_1^*, x_2^*, z) - \delta V_1(z(t))\right] \\ &\leq e^{-\delta t}\left[V_1'(z(t))\left(\dot{z}_{x_1^*} - \dot{z}_{x_1}\right) - u(x_1, z)\right] \\ &= -e^{-\delta t} u(x_1, z); \end{aligned}$$

the last line proceeding from the inequality in equation A.3 and the linear equation of motion, respectively; $\dot{z}_{x_1} \equiv \frac{\partial \dot{z}}{\partial x_1}$, $\dot{z}_{x_1^*} \equiv \frac{\partial \dot{z}}{\partial x_1}|_{x_1 = x_1^*}$. Integrating the first and last elements of this expression over $[0, T]$ produces

$$\int_0^T d\left[e^{-\delta t} V_1(z(t))\right] \leq -\int_0^T e^{-\delta t} u(x_1, z)\, dt.$$

Solving the LHS and taking the lim as $T \to \infty$ then produces

$$V_1(z_0) \geq \int_0^\infty e^{-\delta t} u(x_1, z)\, dt + \lim_{T\to\infty} e^{-\delta T} V_1(z_T).$$

For $x_1^*(z)$ to be a best response to $x_2^*(z)$ it must be that

$$V_1(z_0) \geq \int_0^\infty e^{-\delta t} u(x_1, z)\, dt.$$

The condition ensures this. ■

When the value function associated with $x_2^*()$ satisfies the same conditions then $(x_1^*(), x_2^*())$ support a Nash equilibrium.

Appendix B

No production glut

This appendix examines a version of the game in which the first term in the instantaneous objective function is now logarithmic rather than quadratic (cf. the original instantaneous utility function, equation 2.1):

$$u(x_i, z) = \ln x_i - (z-\zeta)^2, i \in \{1, 2\}.$$

This is explored as an attempt to determine to what extent the LQG results, including the possibility of multiple equilibria, depend on the LGQ game specification.

With linear equation of motion 2.3, the Bellman equation is

$$\delta V_1(z) = \max_{x_1 \geq 0} \left\{ \ln x_1 - (z-\zeta)^2 + V'(z)(x_1 + x_2^* - \beta z) \right\};$$

which has the first order necessary condition

$$x_1^* = \max\left\{0, \frac{-1}{V'(z)}\right\};$$

and yields the differential equations

$$\begin{aligned} \delta W(z) &= \left\{ \begin{array}{l} -\ln[-W'(z)] - (z-\zeta)^2 - 2 - \beta z W'(z) \\ \ln[0] - (z-\zeta)^2 - \beta z W'(z) \end{array} \right\} \\ & \quad \text{when } W'(z) \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} 0; \end{aligned}$$

when play is symmetric and $W(z)$ are the candidate value functions. The corner is always avoided due to the $\ln 0$ term in its payoff.

The interior, defined by $W'(z) < 0$, does not have an explicit solution but a phase diagram may be constructed from

$$w'(z) = \frac{2(z-\zeta) + (\beta+\delta) w(z)}{\frac{1}{w(z)} - \beta z};$$

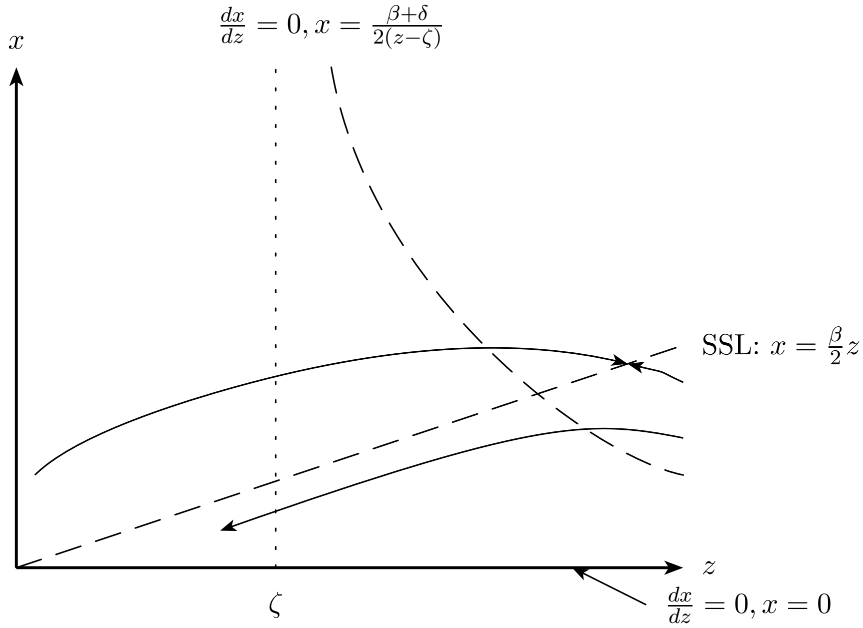

Figure B.1: Logarithmic production felicity

where $w(z) \equiv W'(z)$, or

$$x'(z) = x\frac{\beta+\delta-2(z-\zeta)x}{x+\beta z}.$$

The two loci of points for which $\frac{dx}{dz}=0$ are therefore $x=0$ and $x=\frac{\beta+\delta}{2(z-\zeta)}$. Above the second (in (z,x) space), which asymptotes vertically from $z=\zeta$, $\frac{dx}{dz}<0$. As this locus then continues outside of the feasible state-action space all candidate strategies have positive slope as z approaches zero. The non-invertible locus, for which $\frac{dx}{dz}=\pm\infty$, is also outside the feasible space.

Note that those strategies that remain below the SSL may be discarded: a deviation, for example, to the SSL, would be preferable to falling below $z=\zeta$ (and suffering climate loss) while also reducing production.

This game yields a simpler phase diagram than the original: compare Figure B.1 to Figure 2.2. Again, a continuum of MPE exist. Lemma A.1 holds as candidate strategies converge to the SSL as $T\to\infty$, where they yield finite payoffs. The same problem as was encountered in the linear-quadratic game is encountered when testing Theorem A.2: the unmodified instantaneous utility function is unbounded below. Again, though, a $\underline{u}$ can be defined below the lower bound of the lowest instantaneous payoff to the candidate strategy. One difference between the two sets of candidate strategies is that, in the linear-quadratic game, their intersections with $z=0$ are bounded below by $x=0$ and above by $\hat{x}^b$; here the candidate strategies are not bounded above at $z=0$. This difference does not seem to affect the MPE set.

Appendix C

The asymmetric corner candidate

The techniques used above depend on analysis of a single ordinary differential equation, for which analytical solutions are sometimes possible. When agents behave asymetrically, the game's evolution is governed by two ordinary differential equations, for which an analytical solution is less likely. Certain trivial cases, such as those when one agent plays in the corner, may, however, be examined without new techniques. This is done here, first in the context of two symmetric agents and then in that of asymmetric agents.

C.1 Symmetric agents

Consider the model outlined in Section 2.2. Set agent 2's play to $x_2 = 0$. Agent 1's best response to this, x_1^*, is calculated; it is then asked whether 2's best response to x_1^* is $x_2^* = 0$. Note that $x_1^*(z) = x_2^*(z) = 0$ has been ruled out in Lemma 2.7, above.

Lemma C.1 *The unique best response to $x_2 = 0$ in the model of Section 2.2 is $x_1^* = \hat{x}_1^b$, defined in equation C.2.*

Proof. Agent 1 solves

$$\max_{x_1 \geq 0} \int_0^\infty e^{-\delta t} u(x_1, z)\, dt | \dot{z} = x_1 - \beta z, z(0) = z.$$

This generates, by either Hamilton or Bellman, the differential equation for the interior portion of the solution

$$\frac{dx_1}{dz} = \frac{(\beta + \delta)(x_1 - \xi) + \nu (z - \zeta)}{x_1 - \beta z}.$$

There are singular and general solutions to this; the first are

$$\begin{aligned} x_1^a &= \hat{a} + \hat{s}_a \left(z - \hat{b}\right); \text{ and} \\ x_1^b &= \hat{a} + \hat{s}_b \left(z - \hat{b}\right); \end{aligned} \tag{C.1}$$

where

$$\begin{aligned} \hat{a} &\equiv \beta\hat{b} > 0; \text{ and} \\ \hat{b} &\equiv \frac{(\beta+\delta)\,\xi + \nu\zeta}{\beta\,(\beta+\delta)+\nu} > 0; \\ \{\hat{s}_a, \hat{s}_b\} &= \frac{1}{2}\left[2\beta + \delta \pm \sqrt{(2\beta+\delta)^2 + 4\nu}\right]; \end{aligned}$$

such that $\hat{s}_a > 0 > \hat{s}_b$.

The general solution to the differential equation is

$$K = \left|x_1 - \beta\hat{b} - \hat{s}_a\left(z - \hat{b}\right)\right|^{\hat{\gamma}_1} \left|x_1 - \beta\hat{b} - \hat{s}_b\left(z - \hat{b}\right)\right|^{\hat{\gamma}_2};$$

where K is a (real) constant of integration and

$$\begin{aligned} \hat{\gamma}_1 &\equiv \frac{\delta}{2\,(\hat{s}_b - \hat{s}_a)} - \frac{1}{2}; \\ \hat{\gamma}_2 &\equiv \frac{-\delta}{2\,(\hat{s}_b - \hat{s}_a)} - \frac{1}{2}. \end{aligned}$$

As before, all candidates are eliminated except the analogue of the linear downward sloping candidate,

$$\hat{x}_1^b = \max\left\{0, x_1^b\right\}; \tag{C.2}$$

and those $\hat{x}^3$ strategies for which $\hat{x}^3(0) > 0$. This eliminates all $\hat{x}^3$ when $\hat{x}^a(0) < 0$, a condition that always holds because:

$$\hat{x}^a(0) < 0 \Rightarrow \hat{a} - \hat{s}_a\hat{b} < 0 \Rightarrow (\beta - \hat{s}_a)\,\hat{b} < 0 \Rightarrow \beta < \hat{s}_a;$$

where the last implication is derived from $\hat{b} > 0$. The result follows. ■

As agent 1 was maximising a concave objective function with linear constraints, it is reassuring to find a unique solution.

Now:

Conjecture C.2 *Agent 2's best response to $\hat{x}_1^b$ is not $x_2^* = 0$.*

This is a conjecture rather than a theorem as inequality C.4, below, has not yet yielded an analytical result. While numerical methods are unable to overturn it, this is not a proof.

Agent 2 plays according to

$$\delta V_2(z) = -(x_2^* - \xi)^2 - \nu(z-\zeta)^2 + V'(z)\left((\beta - \hat{s}_b)\left(\hat{b} - z\right) + x_2^*\right); \quad (C.3)$$

where

$$x_2^* = \max\left\{0, \xi + \frac{1}{2}V_2'(z)\right\}.$$

The corner solution requires $x_2^* = x_2^{*\prime} = 0$ and $V_2'(z) \leq -2\xi$. Substituting these requirements into equation C.3 and differentiating yields

$$v'(z) = \frac{(\beta + \delta - \hat{s}_b)\, v(z) + 2\nu(z-\zeta)}{(\beta - \hat{s}_b)\left(\hat{b} - z\right)};$$

when $v(z) \equiv V_2'(z)$. This has solution

$$\begin{aligned} v(z) &= \hat{c}\left(\hat{b} - z\right)^{-\frac{\beta+\delta-\hat{s}_b}{\beta-\hat{s}_b}} \\ &\quad +2\nu\left[\frac{-z}{2\beta+\delta-2\hat{s}_b} + \frac{\zeta}{\beta+\delta-\hat{s}_b} - \frac{(\beta+\hat{s}_b)\,\hat{b}}{(2\beta+\delta-2\hat{s}_b)(\beta+\delta-\hat{s}_b)}\right]; \end{aligned}$$

where $\hat{c}$ is an arbitrary constant of integration. As all $\hat{c} \neq 0$ cause $v(z)$ to become unbounded as $z \to \hat{b}$, a piecewise continuous value function (assumption **C1**) requires that $\hat{c} = 0$. This requirement yields a unique downward sloping linear strategy. As the corner solution must satisfy $v(z) \leq -2\xi$, a necessary and sufficient condition for it to hold generally is that it hold at $z = 0$. It holds, and $x_2^*(z) = 0$ is a best response, when

$$\nu\left[\hat{b}\frac{\beta+\hat{s}_b}{2(\beta-\hat{s}_b)+\delta} - \zeta\right] \geq (\beta+\delta-\hat{s}_b)\,\xi. \quad (C.4)$$

As this inequality has not yielded to analysis, numerical techniques have been attempted: an example of non-negative parameters that allow it to hold has yet to be found. The NAG non-linear maximisation routine, `e04jbc`, can do no better than cause it to hold with equality; this it does by setting three of the parameters to zero.[1]

There are sufficient conditions which guarantee that the conjecture holds. For example:

Lemma C.3 *Agent 2's best response to $\hat{x}_1^b$ is not $x_2^*(z) = 0$ when $\xi \leq \beta\zeta$.*

[1] The starting values selected were $(\beta, \delta, \nu, \xi, \zeta) = \left(\frac{1}{115}, \frac{1}{100}, 5.4 \times 10^{-7}, 1.5, 760\right)$; by completion after the fourth iteration these had been updated to $(\beta, \delta, \nu, \xi, \zeta) = \left(2.145 \times 10^{-5}, 0, 0, 0, 470.84\right)$. The variables used in the routine were rescaled (so that the initial values ranged between .54 and 1.5) to improve conditioning. Subsequent attempts at numerical integration (see Chapter 3) fail for these parameters.

Proof. As the right hand side of inequality C.4 is positive for positive parameters, a non-positive left hand side is sufficient to discard $x_2^*(z) = 0$. The coefficient of $\hat{b}$ in the inequality is less than unity so that $\hat{b} \leq \zeta$ is sufficient to accomplish this. The result follows by expansion of $\hat{b}$. ■

An intuition may be offered for this sufficient condition. In (x_1, z) space it may be seen that $\hat{b} < z < \zeta$ allows agent 2 to deviate to a slightly positive x_2 and receive gains in both production and climate felicity.

C.2 Asymmetric agents

Modify equation 2.1 so that agents' instantaneous utility functions are

$$u_i(x_i, z) = -(x_i - \xi_i)^2 - \nu_i (z - \zeta_i)^2;$$

and their discount rates are $\delta_i \in \Re_{++}$, $i = \{1, 2\}$. The above calculations may be replicated when agent 2 is assumed to play $x_2 = 0$. Now agent 1's best response is $\check{x}_1(z) \equiv \beta\check{b} + \check{s}_b\left(z - \check{b}\right)$ where

$$\begin{aligned}\check{b} &\equiv \frac{\nu_1\zeta_1 + \xi_1(\beta + \delta_1)}{\nu_1 + \beta(\beta + \delta_1)} > 0; \text{ and}\\ \check{s}_b &\equiv \frac{1}{2}[2\beta + \delta_1] - \sqrt{(2\beta + \delta_1)^2 + 4\nu_1}.\end{aligned}$$

The solution to agent 2's Bellman equation when $x_2 = 0$ is

$$\begin{aligned}&v_2(z)\\ = \;& 2\nu_2\left[\frac{\zeta_2}{\beta + \delta_2 - \check{s}_b} - \frac{z}{2\beta + \delta_2 - 2\check{s}_b} - \frac{\check{b}(\beta + \hat{s}_{b1})}{(2\beta + \delta_2 - 2\check{s}_b)(\beta + \delta_2 - \check{s}_b)}\right].\end{aligned}$$

This is possible when

$$\nu_2\left[\check{b}\frac{\beta + \check{s}_b}{2(\beta - \check{s}_b) + \delta_2} - \zeta_2\right] \geq (\beta + \delta_2 - \check{s}_b)\,\xi_2; \qquad \text{(C.5)}$$

a generalisation of inequality C.4.

Conjecture C.4 *Agent 2's best response to $\check{x}_1$ is not $x_2^* = 0$.*

Again, no analytical results are derived from inequality C.5; numerical maximisation, as above, can only cause the inequality to hold with equality. This requires setting four of the parameters to zero.[2]

[2] Nine initial calibrations were selected at random from $(0, 1)$ (the rescaling aiding conditioning). All successful runs converged by the fourth iteration, which set $\delta_2 = \nu_2 = \xi_2 = 0$ in all cases. The other parameter values were not similar.

Chapter 3

Two symmetric agents and asymmetric play

3.1 Introduction

This chapter extends the model of Chapter 2 by allowing symmetric agents to play asymmetrically. This extension is unprecedented in the economics literature, possibly for an obvious reason: it is unclear why agents that are otherwise symmetric would then differ in their play. The answer given in this chapter is that the fiction of allowing symmetric agents to differ in their play simplifies analysis while introducing all the numerical techniques required for the fully asymmetric analysis of Chapter 4.

A second advantage of working with symmetric agents is that the model becomes a supermodel of that explored in Chapter 2, allowing results to be checked against the known case of symmetric play for symmetric agents. To facilitate comparison, a linear-quadratic game is again used here. These are known to have a Markov Perfect Equilibrium in linear strategies, even in the case of asymmetric agents. Uniqueness results for these linear strategies are presented in [BO99, p.324] and [Loc96]. The techniques developed here, though, are applicable to more general differential games as well.

Technically, the difference between this chapter and the previous is that the previous single ordinary differential equation is now replaced by a system of two ordinary differential equations. As systems of differential equations are less likely to yield analytical solutions than are single ordinary differential equations, numerical techniques are adopted here. While lacking the clear vistas of insight provided by analytical solutions, numerical techniques allow solution of more complicated problems, such as those with non-linear equations of motion or asymmetric agents.

Against this advantage, the numerical techniques implemented here are unable to consider discontinuous strategies. Dockner and Sorger's model [DS96] with discontinuous MPE strategies shows that such strategies are certainly possible and should not be dismissed *a priori*. Their model differs from the present, though, in at least two important ways: there is no glut point in consumption and increases in agents' controls decrease the state variable. Both of these features are required for their discontinuous MPE strategies, suggesting that this chapter's failure to consider discontinuous strategies may not be crippling.

Numerical analysis proceeds by integrating over the state space from a grid of initial conditions. The conditions of Chapter 2 and Appendix A are then applied to rule out strategies, refining the candidate set.

In general, a grid of initial conditions will not find isolated MPE strategies, such as the linear $\hat{x}^b$ from Chapter 2, unless it is adapted to finding them. In an attempt to find analogues of $\hat{x}^b$, an adapted second grid of initial conditions is therefore developed, based on the following observation: $\hat{x}^b$ was identified as a singular solution because it and x^a, unique among the solutions to Chapter 2's differential equation, intersected each other. As this plants the suspicion that there may be a relationship between MPE and singularities, the singularity locus is identified for this case of asymmetric play and the strategies through it calculated. In this case, the locus is based on a conic section. The relative ease of its analysis explains some of the appeal of this intermediate chapter: in the case of asymmetric agents, the locus is much more complicated and intuition more difficult.

Section 3.2 presents the new model. Section 3.3 contains definitions and conditions required to test whether a solution to the differential equation system is an MPE. Section 3.4 examines the singularity locus while Section 3.5 discusses the coding and execution. Section 3.6 presents the results of the numerical analysis and Section 3.7 concludes.

3.2 Asymmetric play in the linear-quadratic model

Draw upon the linear-quadratic model of Chapter 2 to consider symmetric agents that may play asymmetrically. As then, differential equations are derived from Bellman's equation.

Again use the linear equation of motion 2.3, reproduced here:

$$\dot{z}(t) = x_1(z) + x_2(z) - \beta z(t).$$

Let the instantaneous utility functions be the quadratic loss functions

$$u(x_i, z) = -(x_i - \xi)^2 - \nu (z - \zeta)^2; \tag{3.1}$$

The Bellman equations are then of the form

$$\delta V_i(z) = \max_{x_i \geq 0} \left\{ -(x_i - \xi)^2 - \nu (z - \zeta)^2 + V_i'(z)(x_1 + x_2 - \beta z) \right\},\ i = 1, 2; \tag{3.2}$$

when the value function is piecewise $\mathcal{C}^1$; δ is the discount rate.

The first order conditions of the Bellman are

$$x_i^* = \max\left\{0, \xi + \frac{V_i'(z)}{2}\right\}, \ i = 1, 2. \tag{3.3}$$

As the objective functions are concave, x_i^* is unique and a maximiser; substitute it into equation 3.2 for

$$\delta V_i(z) = -\left(x_i^* - \xi\right)^2 - \nu\left(z - \zeta\right)^2 + V_i'(z)\left(x_i^* + x_{-i} - \beta z\right), \ i = 1, 2. \tag{3.4}$$

Although x_i^* is unique, solutions to differential equation 3.4 will not be. Therefore, solutions to differential equation 3.4 are not generally solutions to maximisation problem 3.2. Denote a solution to equation 3.4 by $W_i()$; call this a candidate value function; let $\mathcal{W}_i$ be the family of solutions to equation 3.4. Therefore $V_i \in \mathcal{W}_i$.

When the candidate value function is twice differentiable, differentiating differential equation 3.4 with respect to z yields

$$\begin{aligned} \delta W_i'(z) = \ & -2\left(x_i^* - \xi\right)\tfrac{dx_i^*}{dz} - 2\nu\left(z - \zeta\right) + W_i''(z)\left(x_1^* + x_2^* - \beta z\right) \\ & + W_i'(z)\left(\tfrac{dx_1^*}{dz} + \tfrac{dx_2^*}{dz} - \beta\right), \ i = 1, 2. \end{aligned}$$

The following assumes this second differentiation to be valid; points at which it is not will be identified as the scenarios defined below are explored numerically.

Defining $w_i \equiv W_i'$ and tidying produces

$$\begin{aligned} & w_i'(z)\left(x_1^* + x_2^* - \beta z\right) - 2x_i^{*\prime}\left(x_i^* - \xi\right) \\ & = \left(\beta + \delta - x_1^{*\prime} - x_2^{*\prime}\right) w_i(z) + 2\nu\left(z - \zeta\right), i = 1, 2. \end{aligned} \tag{3.5}$$

As either $x_i^* > 0$, the interior of the action space, or $x_i^* = 0$, the action space's corner, there are three possible scenarios for each $z \in Z$: both agents play in the interior, both play on the corner, or one plays in the interior and the other on the corner. These are now explored below.

3.2.1 Both agents interior

In this case, $x_i^* > 0 \forall i = 1, 2 \Rightarrow x_i^* = \xi + \frac{1}{2}w_i(z), x_i^{*\prime} = \frac{1}{2}w_i'$ and equations 3.5 become the differential equation system

$$\begin{bmatrix} g & \frac{1}{2}w_1 \\ \frac{1}{2}w_2 & g \end{bmatrix}\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} \left(\beta + \delta\right) w_1 + 2\nu\left(z - \zeta\right) \\ \left(\beta + \delta\right) w_2 + 2\nu\left(z - \zeta\right) \end{bmatrix}; \tag{3.6}$$

where

$$g \equiv \frac{1}{2}\left(w_1 + w_2\right) - \beta z + 2\xi; \tag{3.7}$$

so that $\dot{z} = g(\mathbf{w}, z)$. A special case of this system is

$$w'(z) = \frac{(\beta + \delta) w + 2\nu (z - \zeta)}{\frac{3}{2}w - \beta z + 2\xi}; \tag{3.8}$$

the single differential equation of symmetric play ($w \equiv w_1 = w_2$) analysed in Chapter 2. A zero in the denominator of this differential equation implies a singular coefficient matrix on the left hand side of system 3.6.

3.2.2 One agent interior, the other cornered

Assume without loss of generality that agent i has cornered. Now $x_j^* > 0, x_i^* = 0 \Rightarrow x_j^* = \xi + \frac{1}{2}w_j(z), x_j^{*\prime} = \frac{1}{2}w_j'; x_i^{*\prime} = 0$ so that equations 3.5 produce the differential equation system

$$\begin{bmatrix} h & 0 \\ \frac{1}{2}w_i & h \end{bmatrix} \begin{bmatrix} w_j' \\ w_i' \end{bmatrix} = \begin{bmatrix} (\beta + \delta) w_j + 2\nu (z - \zeta) \\ (\beta + \delta) w_i + 2\nu (z - \zeta) \end{bmatrix}; \tag{3.9}$$

where

$$h \equiv \frac{1}{2}w_j - \beta z + \xi; \tag{3.10}$$

so that $\dot{z} = h(\mathbf{w}, z)$.

As the first equation in system 3.9 is independent of w_i and of similar form to equation 3.8 it is similarly solvable. This yields the linear solutions

$$\begin{aligned} w_{j,c} &= c + s_c (z - d); \\ w_{j,d} &= c + s_d (z - d); \end{aligned} \tag{3.11}$$

where

$$c \equiv 2\nu \frac{\beta\zeta - \xi}{\beta(\beta\delta) + \nu}; \tag{3.12}$$

$$d \equiv \frac{\xi(\beta + \delta) + \nu\zeta}{\beta(\beta + \delta) + \nu} > 0; \tag{3.13}$$

$$\{s_c, s_d\} \equiv \delta \pm \sqrt{\delta^2 + 4\nu}, s_c > 0 > s_d.$$

The family of interior solutions is therefore

$$K = |w_j - c - s_c (z - d)|^{\gamma_1} |w_j - c - s_d (z - d)|^{\gamma_2};$$

where

$$\begin{aligned} \gamma_1 &= -\frac{2\beta - s_c}{s_d - s_c}; \\ \gamma_2 &= \frac{2\beta - s_d}{s_d - s_c} < 0. \end{aligned}$$

The exponents sum to -1 and the linear solutions to equations 3.11 correspond to the special case of $K = 0$.

Solving the first equation in system 3.9 does not seem to allow an analytical solution for $w_i(z)$.

3.2.3 Both agents cornered

When both agents play on the corner $x_i^* = 0 \forall i = 1,2 \Rightarrow x_i^{*\prime} = 0$ and equations 3.5 produce the differential equation system

$$w_i' = -\frac{(\beta+\delta)\,w_i + 2\nu\,(z-\zeta)}{\beta z}, i = 1,2;$$

whose solution is

$$w_i\,(z) = K_i z^{-\frac{\beta+\delta}{\beta}} + 2\nu\left(\frac{\zeta}{\beta+\delta} - \frac{z}{2\beta+\delta}\right), i = 1,2; \qquad (3.14)$$

where K_i is a constant of integration.

The following lemma provides a sufficient condition for the system to remain in the cornered scenario once reaching it:

Lemma 3.1 *Let $\tilde{z}$ be the least z satisfying $x_i\,(z) = 0$ and $\hat{z} > \tilde{z}$ that satisfying $x_j\,(z) = 0$. A sufficient condition for $x_i\,(z) = x_j\,(z) = 0 \forall z > \hat{z}$ is that*

$$w_i\,(\hat{z}) \geq \frac{2\nu}{\beta+\delta}\,(\zeta - \hat{z})\,.$$

Proof. At $\hat{z}, w_i\,(\hat{z}) \leq w_j\,(\hat{z})$. By equation 3.14, then, $K_i \leq K_j$. For the system to remain cornered it is sufficient that $w_i'\,(z)\,, w_j'\,(z) \leq 0 \forall z > \hat{z}$. Differentiation of equation 3.14 converts this requirement into

$$K_j \geq K_i \geq -\frac{2\beta\nu}{(\beta+\delta)\,(2\beta+\delta)} z^{\frac{2\beta+\delta}{\beta}} \leq 0, \forall z > \hat{z}.$$

The inequality in K_j is thus automatic if that in K_i holds. That in K_i holds if it holds at $z = \hat{z}$ as K_i is fixed but the RHS decreases in z. Isolating K_i (as determined at $\hat{z}$) in equation 3.14 and substituting into the inequality produces

$$\left[w_i\,(\hat{z}) - 2\nu\left(\frac{\zeta}{\beta+\delta} - \frac{\hat{z}}{2\beta+\delta}\right)\right]\hat{z}^{\frac{\beta+\delta}{\beta}} \geq -\frac{2\beta\nu}{(\beta+\delta)\,(2\beta+\delta)}\hat{z}^{\frac{2\beta+\delta}{\beta}}.$$

Some manipulation produces the result. ∎

3.3 Conditions for MPE

As noted above, solutions to ordinary differential equation systems 3.6, 3.9 and 3.14 are generally non-unique and generally do not support MPE. This section therefore establishes conditions for the rejection or acceptance of a particular solution to the differential equations. The latter conditions translate directly from Appendix A; the former are based on those seen in Chapter 2.

First, some terminology is defined.

Definition 3.2 *A* system of differential equations *is*

$$\mathbf{A}(\mathbf{s})\,\mathbf{s}' = \mathbf{f}(\mathbf{s})\,; \tag{3.15}$$

where $\mathbf{s}$ *is an n-vector dependent on its* n^{th} *element, the independent state variable;* $\mathbf{A}(\cdot)$ *is an* $n \times n$ *matrix,* $\mathbf{s}' \equiv \left[\frac{ds_1}{ds_n}, \ldots, \frac{ds_n}{ds_n}\right]'$ *the n-vector of derivatives and* $\mathbf{f}(\cdot)$ *is an n-vector.*

As the elements of $\mathbf{s}$ may be discontinuous, a system of differential equations may represent multiple regimes (i.e. systems 3.6, 3.9 and 3.14). The elements of $\mathbf{A}$ and $\mathbf{f}$ may not be continuous at the points of transition between regimes.

In what follows two interpretations of $\mathbf{s}$ will be used. In one,

$$\mathbf{s} = (w_1(z), w_2(z), z)'\,;$$

while, in the second,

$$\mathbf{s} = (w_1(z(t)), w_2(z(t)), z(t), t)'\,.$$

This latter, more complicated interpretation is used in Section 3.4 as systems 3.6, 3.9 and 3.14 are autonomous in t but not in z. Autonomy in t allows Taylor expansion about $t_0 = 0$, simplifying many equations without loss of generality.

Definition 3.3 *A* path*,* $\mathbf{s}$*, is a solution to system 3.15.*

Definition 3.4 *The point* $\sigma = (\sigma_1, \ldots, \sigma_n)$ lies on path $\mathbf{s}$ *if* $\mathbf{s} = \sigma$ *for some* s_n *in the state's domain.*

Three types of points are of interest here. The first are points at which a path ceases to be a function by 'doubling back' through the domain; these are called non-invertible. Singular points have more than one path through them. Regular points are the rest:

Definition 3.5 *The point* σ *is*

1. *a* non-invertible point *of system 3.15 if it lies on a path* $\mathbf{s}$ *such that:*

   (a)

   $$\left.\frac{\partial s_i}{\partial s_n}\right|_{\sigma} = \infty$$

   for some i; and

   (b) $\exists \delta > 0$ *s.t.,* $\forall \sigma_n + \varepsilon$ *or* $\forall \sigma_n - \varepsilon$*, path* $\mathbf{s}$ *is not defined,* $\delta > \varepsilon > 0$*.*

2. *a* singular point *of system 3.15 if it lies on at least two distinct paths,* $\mathbf{s}$ *and* $\hat{\mathbf{s}}$*;*

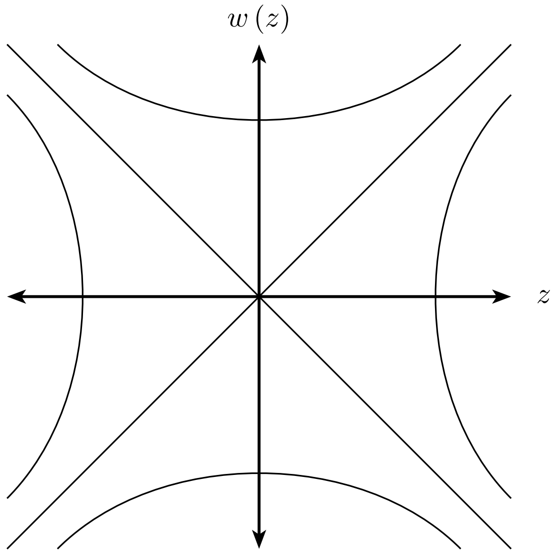

Figure 3.1: A simple example of a singularity at the origin: $w\frac{dw}{dz} = z$.

3. *a* regular point *of system 3.15 otherwise.*

These definitions are illustrated in the example plotted in Figure 3.1. If z is the state variable, the non-invertible points are all $(w = 0, z \neq 0)$, the singular point $(w = 0, z = 0)$, and the regular points all $(w \neq 0, z)$.

Refine the non-invertible points further:

Definition 3.6 *A point σ is a* truly non-invertible *point of system 3.15 if it is non-invertible and if* $\det(\mathbf{A}(\sigma)) = 0$*. The path on which σ lies is then truly non-invertible as well.*

Definition 3.7 *A point σ is a* quasi-non-invertible *point of system 3.15 if it is non-invertible but not truly so. The path on which σ lies is then quasi-non-invertible as well.*

In the example in Figure 3.1 all the non-invertible points are truly so. Quasi-non-invertible points are kinks; the examples of these seen to date mark passage between regimes.

Definition 3.8 *A* candidate MPE strategy *is a path* $\mathbf{s}$ *where:*

1. $\mathbf{w} = (s_1, s_2)'$ *and* $z = s_3$*;*
2. $\mathbf{w}(z)$ *is a function mapping from* Z *to* $\Re^2$*;*
3. $\mathbf{A}$ *and* $\mathbf{f}$ *are defined according to systems 3.6, 3.9 and 3.14 as appropriate.*

3.3.1 Sufficient conditions to disqualify candidate strategies

Lemma 3.9 *Candidate MPE strategies containing non-invertible points cannot be considered as MPE strategies.*

This follows directly from the requirement that a candidate strategy be a function and that it be defined for the whole domain, Z. Otherwise, other agents cannot otherwise form conjectures about an agent's play under all possible circumstances and, therefore, cannot form a best response.

Lemma 3.10 *Candidate MPE strategies that set* $W_i(z) > 0$ *for any* $z \in Z$ *cannot be considered as MPE strategies.*

Proof. As payoff function 3.1 is bounded above by zero for all z, so is the value function, $V_i()$. Any candidate exceeding this cannot be a value function. ■

For reasons that will become apparent in the numerical analysis, it is also important to eliminate strategy pairs for which $(x_1(0), x_2(0)) = \mathbf{0}$.

Lemma 3.11 *Strategy pairs* $(x_1(), x_2())$ *setting* $(x_1(0), x_2(0)) = \mathbf{0}$ *cannot be considered as MPE strategies.*

This generalises Lemma 2.7 to situations in which x_1 and x_2 can differ.

Proof. Assume that the $x_i^*, i \in \{1, 2\}$ that maximises the RHS of Bellman equation 3.2 is zero at $z = 0$. The ensuing differential equation is then

$$W_i'(z) = -\frac{\delta W_i(z) + \xi^2 + \nu(z-\zeta)^2}{\beta z}; \tag{3.16}$$

with solution

$$W_i(z) = \frac{C_i}{z^{\frac{\delta}{\beta}}} - \frac{\nu}{2\beta+\delta}z^2 + \frac{2\nu\zeta}{\beta+\delta}z - \frac{\xi^2+\nu\zeta^2}{\delta}; \tag{3.17}$$

where C_i is a constant of integration. The proof proceeds to rule out all possible values of C_i.

First consider $C_i > 0$. In this case, $W_i(0) = \infty$, violating Lemma 3.10.

Now consider $C_i < 0$. For $x_i^* = 0$ it must be that $W_i'(z) \leq -2\xi$ which, with equation 3.16, yields

$$\delta W_i(z) + \xi^2 + \nu(z-\zeta)^2 \geq 2\beta\xi z.$$

Replacing the $W_i(z)$ term with that in equation 3.17 allows manipulation for

$$\frac{2\beta\nu}{2\beta+\delta}z^{\frac{2\beta+\delta}{\beta}} + \delta C_i \geq 2\beta\left(\xi + \frac{\nu\zeta}{\beta+\delta}\right)z^{\frac{\beta+\delta}{\beta}}.$$

This inequality fails at $z = 0$ for $C_i < 0$, eliminating this case as well.

Consider, finally, $C_i = 0$. Differentiating equation 3.17 with respect to z and requiring that $W_i'(z) \leq -2\xi$ produces, with some manipulation,

$$\frac{(\beta+\delta)\,\xi+\nu\zeta}{\beta+\delta} \leq \frac{\nu}{2\beta+\delta}z;$$

which also fails at $z = 0$. ■

3.3.2 Necessary and sufficient conditions for MPE

Until now discussion has focussed on conditions sufficient to disqualify paths from consideration as MPE strategies. The conditions for doing the reverse have already been presented in Appendix A to Chapter 2.

3.4 Singularities

Singularities are generally of interest in understanding dynamic systems. In the present case, there is a singularity in the solution to symmetric equation 3.8. Furthermore, the unique symmetric linear MPE passes through it. As the numerical numerical techniques used here are based on grids of initial conditions, they tend to find regions rather than isolated paths like those through the singularity. Direct calculation of the singularity locus therefore allows points on it to be used as a second grid of initial conditions. Integration from these then allows exploration of the isolated paths passing through the locus.

Sufficient conditions for a locus of points through which two paths pass are first developed. These conditions are then applied to the systems of differential equations 3.6 and 3.9. The first system is found to have a singularity locus based on a conic section, although with points removed. The second does not have a singularity locus. Singularities are not sought in system 3.14 as its explicit solution may be seen not to yield them.

3.4.1 Theory

The following theorem provides sufficient conditions for the existence of a 2-singularity, defined according to:

Definition 3.12 *An* m-singularity *is a singularity lying on exactly $m \geq 2$ distinct paths.*

This section uses the following notation and assumptions:

1. let $\mathbf{s} = \mathbf{s}(t)$ and rewrite the prototypical differential equation 3.15 as

$$\mathbf{A}(\mathbf{s})\,\dot{\mathbf{s}} = \mathbf{f}(\mathbf{s})\,; \tag{3.18}$$

where derivatives of $\mathbf{s}(t)$ with respect to t are $\dot{\mathbf{s}}, \ddot{\mathbf{s}}, \dddot{\mathbf{s}}$ and so on. Therefore the s_n of equation 3.15 becomes t here. Definition 3.5, defining non-invertible, singular and regular points, is not modified.

2. denote singular points by σ.

3. set $t = 0$ at σ; as the system is autonomous in t no generality is lost.

4. define
$$a_{ij}^{(k)} \equiv \frac{\partial}{\partial s_k} a_{ij}|_\sigma \text{ and } f_i^{(k)} \equiv \frac{\partial f_i}{\partial s_k}|_\sigma;$$
where $[a_{ij}] = \mathbf{A}$ and $[f_i] = \mathbf{f}$. (As the specific a_{ij} and f_i explored here are members of $\mathcal{C}^\infty$ this differentiability assumption is not restrictive.)

5. when $rank\,(\mathbf{A}(\mathbf{s})) < n$ let the vector $\mathbf{q} \neq \mathbf{0}$ be a linear combination of the columns of $\mathbf{A}$ so that $\mathbf{A}(\mathbf{s}) \cdot \mathbf{q} = \mathbf{0}$. Under the same circumstances, let $\mathbf{c} \neq \mathbf{0}$ be a linear combination of rows so that $\mathbf{c} \cdot \mathbf{A}(\mathbf{s}) = \mathbf{0}$. When $rank\,(\mathbf{A}(\mathbf{s})) = n-1$, $\mathbf{c}$ and $\mathbf{q}$ are unique up to a scalar multiple; when $\mathbf{A}$ is symmetric as well let $\mathbf{c}' = \mathbf{q}$.

While the following is initially general one might bear in mind the present problem in which $\mathbf{s} = (w_1(z(t)), w_2(z(t)), z(t), t)$.

Then:

Theorem 3.13 *Given system 3.18, in which $a_{ij}, f_i \in \mathcal{C}^1$, sufficient conditions for the point σ to be a 2-singularity are:*

1. *(non-invertibility) $\mathbf{A}(\sigma)$ has rank $n - 1$;*

2. *(spanning) $\mathbf{f}(\sigma) = \mathbf{A}(\sigma) \cdot \mathbf{r}$ for some n -vector $\mathbf{r}$; and*

3. *(roots) the quadratic equation*
$$\begin{aligned} 0 = \; & \lambda^2 \left[\textstyle\sum_{i,j,k} a_{ij}^{(k)} c_i q_j q_k\right] \\ & +\lambda \textstyle\sum_i c_i \left[2 \sum_{j,k} a_{ij}^{(k)} r_j q_k - \sum_k f_i^{(k)} q_k\right] \\ & + \textstyle\sum_i c_i \left[\sum_{j,k} a_{ij}^{(k)} r_j r_k - \sum_k f_i^{(k)} r_k\right]. \end{aligned} \tag{3.19}$$
has exactly two distinct real roots in λ given the vector $\mathbf{r}$ from the spanning condition.

The intuition behind these conditions is illustrated by the earlier example in Figure 3.1, $w \frac{dw}{dz} = z$.The non-invertibility condition ($w = 0$ in the diagram) imposes a barrier, not to paths, but to functions: paths crossing it generally cease to be functions. When spanning holds as well as non-invertibility, at $(w, z) = (0, 0)$ in the diagram, one may imagine a slit in the barrier through which crossing paths remain functions. The roots condition then ensures that the crossing paths are distinct. It may be seen that the roots of equation 3.19 are homogeneous of degree zero in **c** and homogeneous of degree -1 in **q**; they are not homogeneous in **r**.

The theorem is proven by means of two lemmata.

Lemma 3.14 *Paths through the point σ and satisfying the conditions of Theorem 3.13 have one of two slopes.*

Proof. At σ the i^{th} equation of system 3.18 is

$$\sum_{j=1}^{n} a_{ij}\dot{\sigma}_j = f_i. \tag{3.20}$$

As $\mathbf{A}(\sigma)$ is singular, this fails to determine $\dot{\boldsymbol{\sigma}}$. Therefore take advantage of $t_0 = 0$ and Taylor expand the elements of equation 3.20 for

$$\begin{aligned}&\textstyle\sum_{j=1}^{n}\left\{a_{ij}+\sum_k a_{ij}^{(k)}\dot{\sigma}_k t+\frac{1}{2}\left[\sum_{k,l}\frac{\partial a_{ij}^{(k)}}{\partial s_l}\dot{\sigma}_k\dot{\sigma}_l+a_{ij}^{(k)}\ddot{\sigma}_k\right]t^2+\mathcal{O}\left(t^3\right)\right\}\\ &\times\left\{\dot{\sigma}_j+\ddot{\sigma}_j t+\tfrac{1}{2}\dddot{\sigma}_j t^2+\mathcal{O}\left(t^3\right)\right\}\\ &=\left\{f_i+\textstyle\sum_{k=1}^{n} f_i^{(k)}\dot{\sigma}_k t+\frac{1}{2}\left[\sum_{k,l}\frac{\partial f_i^{(j)}}{\partial s_k}\dot{\sigma}_k\dot{\sigma}_l+\sum_j f_i^{(j)}\ddot{\sigma}_j\right]t^2+\mathcal{O}\left(t^3\right)\right\}.\end{aligned} \tag{3.21}$$

Equality of the coefficients of the powers of t therefore produces an infinite number of equations. That in t^0 is simply equation 3.20. As $\mathbf{A}(\sigma)$ has rank $n-1$, $\dot{\sigma}_j$ is non-unique; define it to be

$$\dot{\sigma}_j = r_j + \lambda q_j; \tag{3.22}$$

where λ is a scalar and q_j is the j^{th} component of **q**. At this point any λ satisfies equation 3.22.

By equality of the coefficients of the t^1 terms

$$\sum_j a_{ij}\ddot{\sigma}_j = -\sum_{j,k} a_{ij}^{(k)}\dot{\sigma}_j\dot{\sigma}_k + \sum_k f_i^{(k)}\dot{\sigma}_k. \tag{3.23}$$

As $\mathbf{A}(\sigma)$ is singular, premultiplication by **c** sets the LHS term in equation 3.23 to zero, yielding

$$\sum_{i,j,k} c_i a_{ij}^{(k)}\dot{\sigma}_j\dot{\sigma}_k = \sum_{i,k} c_i f_i^{(k)}\dot{\sigma}_k.$$

Substitute the non-unique $\dot{\sigma}_j = r_j + \lambda q_j$ into this expression for

$$\sum_{i,j,k} c_i a_{ij}^{(k)}\left(r_j+\lambda q_j\right)\left(r_k+\lambda q_k\right) = \sum_{i,k} c_i f_i^{(k)}\left(r_k+\lambda q_k\right); \tag{3.24}$$

a quadratic in λ. As λ's premultipliers are non-singular, this reduces the non-unique $\dot{\sigma}_j$ to no more than two distinct values. The third condition of

Theorem 3.13, on equation 3.24 (= equation 3.19), then ensures that $\dot{\sigma}_j$ has two distinct, real values. ■

As singularity of $\mathbf{A}(\sigma)$ prevented derivation of $\dot{\sigma}$ from equation 3.20, so did it prevent derivation of $\ddot{\sigma}$ from equation 3.23. The next lemma therefore applies the technique used above to derive $\ddot{\sigma}$.

Lemma 3.15 *A path through a point σ, satisfying the conditions of Theorem 3.13, is uniquely identified by its slope at σ.*

Proof. From equation 3.23, singularity of $\mathbf{A}(\sigma)$ allows

$$\ddot{\sigma}_j = p_j + \mu q_j;$$

where μ is a scalar and q_j is again the j^{th} component of $\mathbf{q}$. Equation of the coefficients of the t^2 terms in the Taylor expansion of equation 3.21 yields

$$\begin{aligned}\sum_j a_{ij}\dddot{\sigma}_j = & \sum_{j,k} \frac{\partial f_i^{(j)}}{\partial s_k}\ddot{\sigma}_j\dot{\sigma}_k + \sum_j f_i^{(j)}\ddot{\sigma}_j - \sum_{j,k,l} \frac{\partial a_{ij}^{(k)}}{\partial s_l}\dot{\sigma}_j\dot{\sigma}_k\dot{\sigma}_l \\ & - \sum_{j,k} a_{ij}^{(k)}\dot{\sigma}_j\ddot{\sigma}_k - 2\sum_{j,k} a_{ij}^{(k)}\dot{\sigma}_k\ddot{\sigma}_j.\end{aligned} \quad (3.25)$$

Premultiply by $\mathbf{c}$ as before for

$$\begin{aligned}0 = & \sum_{i,j,k} c_i\frac{\partial f_i^{(j)}}{\partial s_k}\dot{\sigma}_j\dot{\sigma}_k + \sum_{i,j} c_i f_i^{(j)}\left(p_j + \mu q_j\right) - \sum_{i,j,k,l} c_i\frac{\partial a_{ij}^{(k)}}{\partial s_l}\dot{\sigma}_j\dot{\sigma}_k\dot{\sigma}_l \\ & - \sum_{i,j,k} c_i a_{ij}^{(k)}\left(p_k + \mu q_k\right)\dot{\sigma}_j - 2\sum_{i,j,k} c_i a_{ij}^{(k)}\dot{\sigma}_k\left(p_j + \mu q_j\right).\end{aligned} \quad (3.26)$$

As this is linear in μ and as μ's premultipliers are non-singular, there is a unique μ satisfying it. Hence, given $\dot{\sigma}_j$, $\ddot{\sigma}_j$ is unique.

The coefficients of the higher order terms, $t^n, n > 2$, are obtained by further differentiating equation 3.25. As this reveals $\dddot{\sigma}_j$ to be a linear function of the $\ddot{\sigma}$ terms, higher order derivatives, $\frac{d^{n+1}\sigma_j}{dt^{n+1}}$, are also linear in $\frac{d^n\sigma_j}{dt^n}, n > 2$. Thus, given any set of lower order derivatives, $\left\{\dot{\sigma}_j, \ddot{\sigma}_j, \ldots, \frac{d^n\sigma_j}{dt^n}\right\}$, $\frac{d^{n+1}\sigma_j}{dt^{n+1}}$ is unique. ■

Theorem 3.13 avoided discussion of necessary conditions. This is a pragmatic decision as failures of, for example, the rank condition of Theorem 3.13 become quite complicated. Imagine that $rank\left(\mathbf{A}(\sigma)\right) = n - 2$. There are now independent n-vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ such that $\mathbf{c}_1 \cdot \mathbf{A}(\sigma) = \mathbf{c}_2 \cdot \mathbf{A}(\sigma) = \mathbf{0}$. The logic above, which differentiates the general system of differential equations 3.18, produces

$$\begin{aligned}\mathbf{c}_1\left(\mathbf{A}'\dot{s} + \mathbf{A}\ddot{\mathbf{s}}\right) &= \mathbf{c}_1\mathbf{f}'; \text{ and} \\ \mathbf{c}_2\left(\mathbf{A}'\dot{s} + \mathbf{A}\ddot{\mathbf{s}}\right) &= \mathbf{c}_2\mathbf{f}';\end{aligned}$$

or, by the definition of the $\mathbf{c}$ vectors,

$$\begin{aligned}\mathbf{c}_1\mathbf{A}'\dot{s} &= \mathbf{c}_1\mathbf{f}'; \text{ and} \\ \mathbf{c}_2\mathbf{A}'\dot{s} &= \mathbf{c}_2\mathbf{f}'.\end{aligned}$$

If

$$\dot{\sigma} = r_j + \lambda_1 q_{1,j} + \lambda_2 q_{2,j};$$

where $\mathbf{A}(\sigma) \cdot \mathbf{q}_1 = \mathbf{A}(\sigma) \cdot q_2 = \mathbf{0}$ and λ_1 and λ_2 are constants, then each of these defines a conic in λ_1 and λ_2. As both must hold, the acceptable values of λ_1 and λ_2 are those representing the intersection of the conics. As some of these intersections may be in the complex hyperplane, this chapter avoids further exploration of these issues.

3.4.2 Singularities in system 3.6

The preceding allows more careful analysis of the singularities of the system of differential equations 3.6. This section is concerned with the examination of 2-singularities. It first examines the sets of points satisfying each of the conditions of Theorem 3.13 and then assembles them to describe the 2-singularity locus. It will be seen that the 2-singularity locus is a conic section, with some points removed. Some of these removed points are regular rather than singular; others allow the passage of no real paths.

The non-invertibility and spanning conditions

Lemma 3.16 *The points of system 3.6 satisfying the non-invertibility condition of Theorem 3.13 define a cone.*

Proof. In system 3.6, the non-invertibility condition of Theorem 3.13 becomes

$$g^2 = \frac{1}{4} w_1 w_2. \tag{3.27}$$

This surface is the union of the generating lines

$$g = \frac{1}{2} p w_1 = \frac{1}{2} \frac{1}{p} w_2; \tag{3.28}$$

parameterised by the finite $p \neq 0$. The lines in this family are non-parallel, pass through the common $(w_1, w_2, z) = \left(0, 0, \frac{2\xi}{\beta}\right)$ and hence define a cone. ■

Lemma 3.17 *The points of system 3.6 satisfying the non-invertibility and spanning conditions of Theorem 3.13 form a conic section and a line.*

Proof. By condition 2 of Theorem 3.13, the spanning condition is defined at σ. Here $\mathbf{A}$ is non-invertible and, by spanning, its columns are proportional to each other, allowing system 3.6 to be written as

$$\rho \begin{bmatrix} 2g \\ w_2 \end{bmatrix} = (\beta + \delta) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + 2\nu (z - \zeta) \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

where ρ is some scalar. Solving ρ out of the two equations yields

$$(\beta + \delta)(w_1 w_2 - 2gw_2) + 2\nu(z - \zeta)(w_2 - 2g) = 0.$$

With equation 3.27 this produces

$$\begin{aligned}(\beta + \delta)\left(4g^2 - 2gw_2\right) + 2\nu(z - \zeta)(w_2 - 2g) &= 0;\\ (2g - w_2)\left[g(\beta + \delta) - \nu(z - \zeta)\right] &= 0. \end{aligned} \tag{3.29}$$

Each of the terms in equation 3.29 defines a plane. The intersection of the first and the non-invertibility cone of equation 3.27 is the degenerate conic $w_1 = w_2 = 2(\beta z - 2\xi)$, a line. As the second element is the non-degenerate intersection of a plane and the cone, it is a conic section. ■

Lemma 3.18 *Lemma 3.17's conic section has two branches in* (w_1, w_2, z) *space when*

$$\beta\left[\beta(\beta + \delta) + 2\nu\right](\beta + \delta) > 3\nu^2; \tag{3.30}$$

and one otherwise.

Proof. The axis of Lemma 3.17's cone lies on the $w_1 = w_2$ plane. As the conic intersection of the cone with Lemma 3.17's plane is symmetric in w_1 and w_2, whatever branches it has must cross the $w \equiv w_1 = w_2$ plane. On this symmetric plane there are three lines relevant to this proof: the generating lines

$$\begin{aligned} z &= \frac{1}{\beta}\left(\frac{3}{2}w + 2\xi\right) \Leftrightarrow p = -1;\\ z &= \frac{1}{\beta}\left(\frac{1}{2}w + 2\xi\right) \Leftrightarrow p = 1; \end{aligned}$$

and the spanning line when $\mathbf{A}$ is non-invertible:

$$z = \frac{w(\beta + \delta) + 2\xi(\beta + \delta) + \nu\zeta}{\beta(\beta + \delta) + \nu}. \tag{3.31}$$

When the spanning line intersects both generating lines in the same half cone (with vertex at $(w, z) = \left(0, \frac{2\xi}{\beta}\right)$) the conic section has one branch; otherwise it has two. The intersections occur at

$$(w, z) = \left(\frac{2\nu(\beta\zeta - 2\xi)}{\beta(\beta + \delta) + 3\nu}, \frac{2\xi(\beta + \delta) + 3\nu\zeta}{\beta(\beta + \delta) + 3\nu}\right); \tag{3.32}$$

$$(w, z) = \left(\frac{-2\nu(\beta\zeta - 2\xi)}{\beta(\beta + \delta) - \nu}, \frac{2\xi(\beta + \delta) - \nu\zeta}{\beta(\beta + \delta) - \nu}\right). \tag{3.33}$$

They will both be in the same half cone when the product of the intersections in w is positive:

$$\left(\frac{2\nu(\beta\zeta - 2\xi)}{\beta(\beta + \delta) + 3\nu}\right)\left(\frac{-2\nu(\beta\zeta - 2\xi)}{\beta(\beta + \delta) - \nu}\right) > 0.$$

As the numerator of this product is always negative this simplifies to

$$(\beta(\beta+\delta)+3\nu)(\beta(\beta+\delta)-\nu)<0;$$

or

$$\beta[\beta(\beta+\delta)+2\nu](\beta+\delta)<3\nu^2.$$

Otherwise the intersections are in opposite half cones. ■

The roots condition

The roots condition of Theorem 3.13 fails in a simple and specific way, as shown in Lemma 3.19. As failure occurs on the conic section defined by Lemma 3.17, the locus of points of that conic satisfying the conditions of Theorem 3.13 is punctured. The subsequent lemmata then show more general ways in which the roots condition fails.

A symmetric line that satisfies spanning and non-invertibility does so by setting p, defined in equation 3.28 to 1 so that $2g = w_2 = w_1$. Nevertheless, this line through the symmetric plane is not a solution to differential equation 3.8, the symmetric special case of system 3.6:

Lemma 3.19 *The line defined in Lemma 3.17 generally fails to satisfy condition 3 of Theorem 3.13.*

The argument used in the proof is a special case of that in Theorem 3.13. Instead of expanding the general system 3.18, system 3.6 specifically is expanded. The analogue of quadratic equation 3.19 is derived by differentiating twice (the coefficients of t^1) and premultiplying by the vector $\mathbf{c}$.

Proof. Differentiating system 3.6 with respect to z yields the specific form of equation 3.23,

$$\begin{aligned}&\begin{bmatrix} g & \frac{1}{2}w_1 \\ \frac{1}{2}w_2 & g \end{bmatrix}\begin{bmatrix} w_1'' \\ w_2'' \end{bmatrix}+\begin{bmatrix} g' & \frac{1}{2}w_1' \\ \frac{1}{2}w_2' & g' \end{bmatrix}\begin{bmatrix} w_1' \\ w_2' \end{bmatrix}\\ = \;& (\beta+\delta)\begin{bmatrix} w_1' \\ w_2' \end{bmatrix}+2\nu\begin{bmatrix} 1 \\ 1 \end{bmatrix};\end{aligned}$$

where $g' = \frac{1}{2}(w_1' + w_2') - \beta$. When $g = \frac{1}{2}w \equiv \frac{1}{2}w_1 = \frac{1}{2}w_2$, thus satisfying equations 3.27 and 3.29, system 3.6 becomes

$$w_1' + w_2' = \frac{2}{w}[(\beta+\delta)w + 2\nu(z-\zeta)]. \tag{3.34}$$

By substituting out the g' terms in the system of second derivatives this relationship and that between g and w simplify the system to

$$\begin{aligned}&\begin{bmatrix} \frac{1}{2}w & \frac{1}{2}w \\ \frac{1}{2}w & \frac{1}{2}w \end{bmatrix}\begin{bmatrix} w_1'' \\ w_2'' \end{bmatrix}+\begin{bmatrix} \delta+2\nu\frac{z-\zeta}{w} & \frac{1}{2}w_1' \\ \frac{1}{2}w_2' & \delta+2\nu\frac{z-\zeta}{w} \end{bmatrix}\begin{bmatrix} w_1' \\ w_2' \end{bmatrix}\\ = \;& (\beta+\delta)\begin{bmatrix} w_1' \\ w_2' \end{bmatrix}+2\nu\begin{bmatrix} 1 \\ 1 \end{bmatrix}.\end{aligned}$$

The premultiplying vector $(c_1, c_2) = (1, -1)$ cancels the second derivatives to produce

$$\left(\delta + 2\nu\frac{z-\zeta}{w} - \frac{1}{2}w_2'\right) w_1' - \left(\delta + 2\nu\frac{z-\zeta}{w} - \frac{1}{2}w_1'\right) w_2' = (\beta + \delta)\left(w_1' - w_2'\right);$$

or

$$\left(2\nu\frac{z-\zeta}{w} - \beta\right)\left(w_1' - w_2'\right) = 0; \tag{3.35}$$

the specific form of quadratic condition 3.19. With the substitution $w_i' = r_i + \lambda q_i$ and $\mathbf{q} = \mathbf{c}'$, equation 3.35 becomes

$$\left(2\nu\frac{z-\zeta}{w} - \beta\right)(r_1 - r_2 + 2\lambda) = 0;$$

which does not generally have two real, distinct roots in λ. ∎

The preceding lemma dealt with one of two possible cases of symmetric play satisfying non-invertibility, that corresponding to the generating line with parameter $p = 1$. A similar process confirms that a point on the $p = -1$ line, the other case, does satisfy the conditions of Theorem 3.13; this produces the 2-singularity at the intersection of the linear solutions to the symmetric differential equation 3.8, as illustrated in Figure 2.2.

More generally, there are two ways in which the quadratic equation 3.19 will fail to have distinct, real roots. The next two lemmata examine these possibilities. First, the coefficient of the square term may be zero:

Lemma 3.20 *In the case of system 3.6 the coefficient of λ^2 in quadratic equation 3.19 is zero iff $p = 1$.*

Proof. At non-invertibility, the relationship $g = \frac{1}{2}pw_1 = \frac{1}{2}\frac{1}{p}w_2$ allows system 3.6 to be written

$$g\begin{bmatrix} 1 & \frac{1}{p} \\ p & 1 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 2\begin{bmatrix} (\beta+\delta)\frac{1}{p}g + \nu(z-\zeta) \\ (\beta+\delta)pg + \nu(z-\zeta) \end{bmatrix}. \tag{3.36}$$

Vectors that set $\mathbf{A} \cdot \mathbf{q} = \mathbf{0}$ and $\mathbf{c} \cdot \mathbf{A} = \mathbf{0}$ are

$$\mathbf{q} = (1, -p, 0)';$$

and

$$\mathbf{c} = \left(1, -\frac{1}{p}, 0\right).$$

The requirement from equation 3.19 that

$$\sum_{i,j,k} a_{ij}^{(k)} c_i q_j q_k \neq 0;$$

is expanded to

$$\begin{aligned}
& \sum_i c_i \left[\sum_j q_j \sum_k a_{ij}^{(k)} q_k \right] \\
= & \left[\sum_j^2 q_j \left(a_{1j}^{(1)} - p a_{1j}^{(2)} \right) \right] - \frac{1}{p} \left[\sum_j^2 q_j \left(a_{2j}^{(1)} - p a_{2j}^{(2)} \right) \right] \\
= & \left[\left(a_{11}^{(1)} - p a_{11}^{(2)} \right) - p \left(a_{12}^{(1)} - p a_{12}^{(2)} \right) \right] \\
& - \frac{1}{p} \left[\left(a_{21}^{(1)} - p a_{21}^{(2)} \right) - p \left(a_{22}^{(1)} - p a_{22}^{(2)} \right) \right] \\
= & \frac{1}{2} - p - \frac{1}{p} \left[\frac{1}{2} p^2 - p \right] = \frac{3}{2} - \frac{3}{2} p = \frac{3}{2} (1 - p) \neq 0 \\
\Leftrightarrow & \; p \neq 1.
\end{aligned}$$

■

Therefore, given the **c** and **q** used here, the coefficient of λ^2 in this case is $\frac{3}{2}(1-p)$. The $p = 1$ case had already been discarded as a possible singularity in Lemma 3.19.

The second way in which the quadratic equation may fail to have distinct, real roots is by having a negative discriminant:

Lemma 3.21 *Quadratic equation 3.19 has a positive discriminant in the case of system 3.6 iff*

$$\begin{aligned}
& [\beta + \delta]^2 p^4 - \left[\beta^2 + 3\nu + \beta\delta \right] p^3 - \left[2\beta\delta + \delta^2 - 6\,\nu \right] p^2 \\
& - \left[\beta^2 + 3\nu + \beta\delta \right] p + [\beta + \delta]^2 > 0;
\end{aligned} \tag{3.37}$$

when $p \neq 1$.

Proof. See Appendix D. ■

It is not clear how to interpret the failure of condition 3.37: no real paths pass through the points concerned.

As the truth of inequality 3.37 depends on the model's calibration, there are cases when it holds for all p:

Lemma 3.22 *Inequality 3.37 holds for all* p *iff*

$$8\nu > 3\beta^2 + 8\beta\delta + 4\delta^2 + \nu \frac{2\beta(\beta + \delta) + 3\nu}{(\beta + \delta)^2}. \tag{3.38}$$

Proof. As inequality 3.37 holds for $p = 0$ only consider those $p \neq 0$. Divide inequality 3.37 by $p^2 \neq 0$ for

$$(\beta + \delta)^2 \left(p^2 + \frac{1}{p^2} \right) - \left(\beta^2 + 3\nu + \beta\delta \right) \left(p + \frac{1}{p} \right) - \left(2\beta\delta + \delta^2 - 6\nu \right) > 0;$$

or

$$(\beta+\delta)^2\left(\left(p+\tfrac{1}{p}\right)^2-2\right)-\left(\beta^2+3\nu+\beta\delta\right)\left(p+\tfrac{1}{p}\right)$$
$$-\left(2\,\beta\delta+\delta^2-6\nu\right)>0.$$

Define $q\equiv p+\frac{1}{p}$ to reduce this to the quadratic

$$(\beta+\delta)^2\,q^2-\left(\beta^2+3\nu+\beta\delta\right)q-\left(2\beta\delta+\delta^2-6\nu\right)-2\,(\beta+\delta)^2>0. \quad (3.39)$$

As the coefficient of q^2 is positive its stationary points are minima. The stationary points satisfy

$$q^*=\frac{\beta\,(\beta+\delta)+3\nu}{2\,(\beta+\delta)^2}.$$

Substituting this into inequality 3.39 yields

$$-\frac{[\beta\,(\beta+\delta)+3\nu]^2}{4\,(\beta+\delta)^2}-\left(2\beta\delta+\delta^2-6\nu\right)-2\,(\beta+\delta)^2>0;$$

which may be manipulated to produce condition 3.38. When this condition holds, the inequality has no real roots. ■

The 2-singularity locus

The various statements made above may now be assembled into:

Theorem 3.23 *The 2-singularity locus of system 3.6 coincides with the conic section defined in Lemma 3.17 except when:*

1. $p=1$ *(as per Lemma 3.19); or*
2. p *is such that inequality 3.37 fails.*

While the conic section underlying the 2-singularity locus may have one or two branches, the locus never ceases to exist as a result of the auxiliary condition, $w_i\geq-2\xi, i=1,2$:

Lemma 3.24 *The auxiliary condition* $w_i\geq-2\xi, i=1,2$ *cannot remove the entire 2-singularity locus.*

Proof.
Consider the intersection of the non-invertibility line with parameter $p=-1$ and the objects satisfying non-invertibility and spanning in equation 3.29; their intersection satisfies the conditions of Theorem 3.13. This point, identified in equation 3.32, always satisfies $(w,z)>(-2\xi,0)$. Therefore, this part of the planar conic is always a 2-singularity. ■

It may also be of interest to know when the 2-singularity locus has two branches:

Lemma 3.25 *When condition 3.38 holds, the 2-singularity locus in system 3.6 has two branches iff*

$$\frac{\nu(\beta\zeta - 2\xi)}{\beta(\beta+\delta) - \nu} < \xi. \tag{3.40}$$

and inequality 3.30 holds.

Proof. Inequality 3.30 in Lemma 3.18 provided a necessary and sufficient condition for the conic to have two branches. The non-invertibility generating line with parameter $p = 1$ intersects the spanning line in equation 3.31 at $w_1 = w_2 > -2\xi$ iff inequality 3.40 holds. As, from Lemma 3.24, the generating line with parameter $p = -1$ always intersects the spanning line at a $w_1 = w_2 > -2\xi$, the proof follows. ■

As a concluding note, cases in which $rank(\mathbf{A}(\sigma)) < n-2$ have not been examined here. Clearly, cases of $n - d$ *non-invertibility*, where $d > 2$, are impossible in system 3.6 as $n = 2$. The $d = 2$ case is possible but requires that $\mathbf{A}(\sigma) = \mathbf{0}$, hence $w_1 = w_2 = g = 0 \Rightarrow z = \frac{2\xi}{\beta}$; these conditions are only satisfied at $(\mathbf{w}, z) = \left(\mathbf{0}, \frac{2\xi}{\beta}\right)$, the apex of the non-invertibility cone. Spanning would then require that $z = \zeta$ so that singularities in this case would require the parameter restriction $2\xi = \beta\zeta$. Given the non-genericity of this restriction, and the costs of presenting the more general theory to address this case, it is not examined here.

3.4.3 Singularities in system 3.9

As $w_i \leq -2\xi, i = 1, 2$ the minimum $rank(\mathbf{A}(\sigma))$ in system 3.9 is $n - 1$. The worst case, from the point of view of multiple solutions, is that when $rank(\mathbf{A}(\sigma)) = n - 1$ but Theorem 3.13's spanning condition holds (so that there is a solution) along with its roots condition (so that there are two solutions). Therefore the only singularities that need to be considered here are 2-singularities.

Theorem 3.26 *There are no 2-singularities in system 3.9.*

Proof. Non-invertibility requires that $h = 0$, so that $w_j = 2(\beta z - \xi)$. When $h = 0$ the system is

$$\begin{bmatrix} 0 & 0 \\ \frac{1}{2}w_i & 0 \end{bmatrix} \begin{bmatrix} w_j' \\ w_i' \end{bmatrix} = \begin{bmatrix} (\beta+\delta)w_j + 2\nu(z-\zeta) \\ (\beta+\delta)w_i + 2\nu(z-\zeta) \end{bmatrix}.$$

If spanning occurs, the first equation produces

$$w_j = -2\nu\frac{z-\zeta}{\beta+\delta};$$

which, with the above implication of $h = 0$ for w_j and z, allows

$$(w_j, z) = (c, d);$$

where c and d are defined in equations 3.12 and 3.13. Spanning then requires that the second equation satisfy

$$\frac{1}{2}w_i w_j' = (\beta+\delta)\, w_i + 2\nu\,(z-\zeta)\,;$$

or, with the appropriate substitutions,

$$w_j' = 2\,(\beta+\delta)\left[1 + \frac{1}{w_i}\frac{2\nu\,(\xi-\beta\zeta)}{\beta\,(\beta+\delta)+\nu}\right].$$

As $w_i \leq -2\xi$ by construction of the scenario, w_j' is unique. ■

3.5 Coding and execution

This section describes how the insights developed so far are adapted to numerical integration. It first mentions the general technique used for solving the differential equations and then discusses how the initial conditions are derived. The bulk of this section, though, is concerned with a description of the implementation of the various conditions for disqualifying candidate paths developed in Section 3.3. Finally, issues relating to the precision of numerical calculations are addressed.

3.5.1 Solution method

The solution to a differential game is a tuple of value or strategy functions. There is a growing literature on the application of projection methods to functional problems; see Judd [Jud98] for a proper presentation. Instead of solving the infinite dimensional functional problem, these methods solve a finite dimensional approximation. This, then, merely requires determination of a finite number of coefficients. In solving problems involving differential equations, such as optimal growth problems, some method is needed to select the solution to the original problem from those to the differential equations. This may be done by, in the optimal growth example, forcing the solution path to pass through the steady state. The literature suggests that these techniques might be quite rapid [Jud92].

In the present problem the solution paths must satisfy transversality conditions and it is less clear how to select the solution path from the candidates. Consequently, the techniques used here are more traditional finite difference methods. Coding is performed in C for speed; the code is available upon request from the author.

3.5.2 Initial conditions

The finite difference method is implemented by integrating forward in z from a grid of initial conditions, $(w_1(0), w_2(0))$. As the agents are symmetrical, only half of the grid is explored. The upper and lower bounds of the initial conditions grid are somewhat arbitrarily set. The lower bound is usually set at $w_i(0) = -3\xi, i = 1, 2$ as those paths starting from $w_i(0) \leq -2\xi, i = 1, 2$ imply $\mathbf{x}(0) = \mathbf{0}$ and are discarded under Lemma 3.11. Similarly, the upper bound is generally set to $w_i(0) = 0$, implying $x_i(0) = \xi$. This is a reference to the case of symmetric play as $x_i(0) > \xi$ paths there could be discarded for violation of a transversality condition. This upper bound may not be meaningful in the case of asymmetric play.

As unique paths are not generally found by a grid not designed to look for them, the 2-singularity locus of system 3.6 is also computed. Paths are then computed off of points on this locus. Points on the locus satisfy equations 3.27 (non-invertibility) and 3.29 (spanning given non-invertibility). As the possibility that $2g = w_2$ was eliminated by the distinct roots condition in Lemma 3.19, the second condition is reduced to

$$g = \frac{\nu(z - \zeta)}{\beta + \delta}.$$

With g defined in equation 3.7, calculation of the 2-singularity reduces to a problem of three unknowns and three equations (one of them quadratic). Computationally, the 2-singularity locus is then calculated by setting $w_1 = -2\xi$ and using a non-linear solver to determine (w_2, z); w_1 is then varied and the procedure repeated until reaching the singular point on the $w_1 = w_2$ symmetric plane. For the calibrations used here, this technique has sufficed as the 2-singularity locus intersects $w_i = -2\xi$.

3.5.3 Testing paths

Having determined initial conditions, paths are then tested against various conditions as they evolve. Two tests are used to detect truly non-invertible paths. The first is triggered by a sign change in the determinant. As computation may become slow around these points, the second test is triggered when the determinant falls in absolute value to less than some small tolerance.

This second test falsely rejects some genuine functions of the sort illustrated in Figure 3.2. Here, the determinant becomes small as $w \to 0$ but, in all cases, the paths still define functions. None of the paths are non-invertible in the sense of 'doubling back', the sense in which non-invertibility was defined.

Define points of the sort at $w = 0, z \neq 0$ in Figure 3.2 by:

Definition 3.27 *An* inverted inflection point *of a function $w()$ is a point $w(z)$ at which $\frac{dw}{dz} = \pm\infty$ and $\frac{d^2z}{dw^2} = 0$. When $w()$ is a vector, an inverted inflection point requires these relationships to hold for all elements of the vector $w()$.*

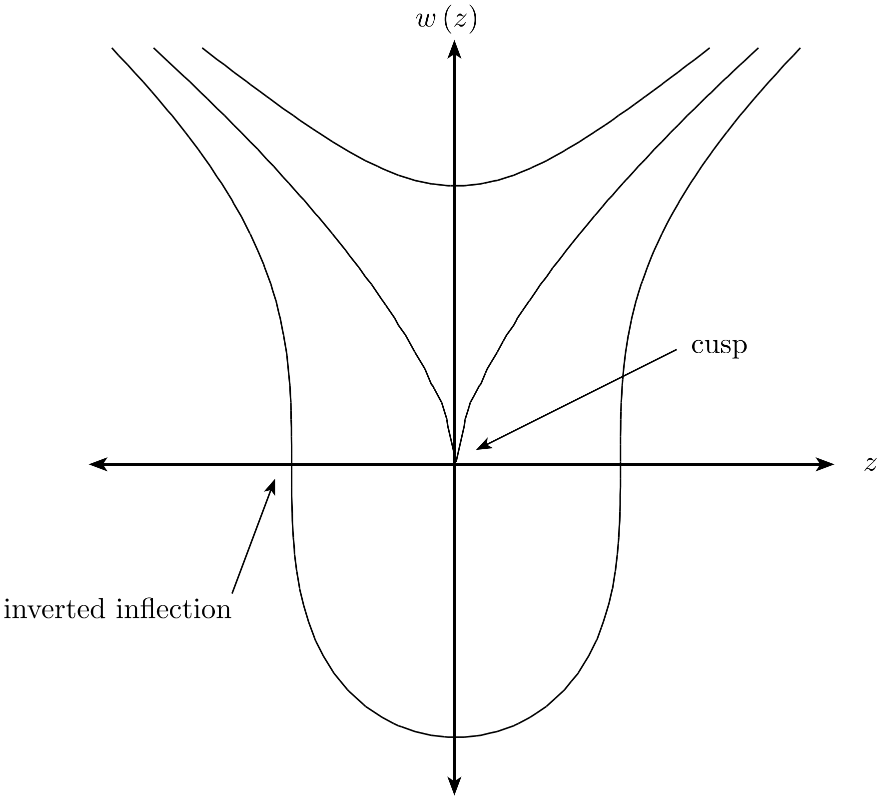

Figure 3.2: Small determinants but still functions: $w^2 \frac{dw}{dz} = z$

Therefore:

Theorem 3.28 *There are no inverted inflection points in system 3.6.*

Proof. In system 3.6, the second condition of the definition of an inverted inflection point implies that

$$\begin{aligned} \frac{d^2 z}{dw_i^2} &= \frac{g - \frac{1}{4}w_j}{(\beta+\delta)\left(g - \frac{1}{2}w_j\right)w_i + 2\nu(z-\zeta)\left(g - \frac{1}{2}w_i\right)} \\ &\quad - \frac{\left(g^2 - \frac{1}{2}w_1 w_2\right)(\beta+\delta)\left[\frac{1}{2}(w_i - w_j) + g\right]}{\left[(\beta+\delta)\left(g - \frac{1}{2}w_j\right)w_i + 2\nu(z-\zeta)\left(g - \frac{1}{2}w_i\right)\right]^2}, i \neq j \in \{1,2\}. \end{aligned}$$

The first condition of the definition holds at the non-invertible points $g^2 = \frac{1}{4}w_1 w_2$. Substitution then produces

$$\frac{d^2 z}{dw_i^2}|_{NI} = \frac{g - \frac{1}{4}w_j}{(\beta+\delta)\left(g - \frac{1}{2}w_j\right)w_i + 2\nu(z-\zeta)\left(g - \frac{1}{2}w_i\right)}, i \neq j \in \{1,2\}.$$

A necessary condition for this to be zero is that $g = \frac{1}{4}w_j$, implying (with the non-invertible condition) that $g = w_i$. Further substitution then allows

$$\frac{d^2 z}{dw_i^2}|_{NI} = \frac{0}{-(\beta+\delta)\, g^2 - \nu\,(z-\zeta)\, g}.$$

Necessary and sufficient conditions for $\frac{d^2 z}{dw_i^2}|_{NI} = 0$ are now that $g = \frac{1}{4} w_j = w_i \neq 0$. As this must hold simultaneously for both $i = 1, 2$ the restriction that $g \neq 0$ prevents acceptable non-invertible paths existing. ■

Theorem 3.29 *There are no inverted inflection points of system 3.9.*

Proof. Now the second condition produces

$$\frac{d^2 z}{dw_j^2} = \frac{\frac{1}{2}}{(\beta+\delta)\, w_j + 2\nu\,(z-\zeta)} - \frac{h\,(\beta+\delta)}{[(\beta+\delta)\, w_j + 2\nu\,(z-\zeta)]^2};$$

for the non-cornered agent. At non-invertibility, $h = 0$ so that

$$\frac{d^2 z}{dw_j^2} = \frac{\frac{1}{2}}{(\beta+\delta)\, w_j + 2\nu\,(z-\zeta)} \neq 0.$$

As there are therefore no invertible inflection points along $w_j\,()$, there are none in system 3.9. ■

Now define points of the sort at $(w, z) = \mathbf{0}$ in Figure 3.2 by:

Definition 3.30 *A* cusp *of a function $w\,()$ is a point $w\,(z)$ at which*

1. *$w\,(z)$ is finite; and*
2. *$w'\,(z)$ is not defined; and*
3. *either* $\lim_{z+} w'\,(z) = \infty$ *and* $\lim_{z-} w'\,(z) = -\infty$*, or vice versa.*

When $w\,()$ is a vector, a cusp requires the foregoing conditions hold for all elements of the vector w.

This is a very particular definition of a cusp, designed to identify points at which $\det(\mathbf{A})$ approaches zero. Notably, the definition excludes kinks (such as $z = 0$ in $w\,(z) = abs\,(z)$) and, by its vertical orientation, many of the usual cusps. The definition also excludes non-invertible points by its requirement that $w\,()$ be a function.

The possibility of a cusp in system 3.6 is addressed by adding a further condition to the code according to the following two observations. First, the cusp's infinite derivatives require that $g^2 = \frac{1}{4} w_1 w_2$ (non-invertibility) to set the denominators of system 3.6 to zero. The second observation requires that the solution path in question be considered as the parameterised curve $(w_1\,(s), w_2\,(s), z\,(s))$, for parameter s. As the cusp is a point where w_i' is not defined, it must have no tangents, thus satisfying $\left(\frac{dw_1(s)}{ds}, \frac{dw_2(s)}{ds}, \frac{dz(s)}{ds}\right) = \mathbf{0}$ at the cusp. A natural parameter to use is $s = t$ in which case this condition may be expressed as

$$\left(\frac{dw_1}{dz}, \frac{dw_2}{dz}, 1\right) \dot{z} = \mathbf{0};$$

for which a necessary and sufficient condition is that $\dot{z} = 0$ or, in the present notation, $g = 0$.

Therefore, when approaching $g^2 = \frac{1}{4} w_1 w_2$ (non-invertibility) in system 3.6, those paths for which $g \approx 0$ (cusp) are identified.

This approach is somewhat unsatisfactory as one might expect paths containing cusps to be isolated, and therefore not usually detected by the present technique of integrating along grids of initial conditions. In the present case of linear terms in the **A** matrix one might expect a formal non-existence proof to be possible.

A stronger statement may be made about system 3.9:

Theorem 3.31 *There are no cusps in system 3.9.*

Proof. The infinite derivatives as a cusp is approached require that $h \to 0 \Rightarrow w_j = 2(\beta z - \xi)$; the cusp also requires a sign change in $w'()$ around it. Consider, then

$$w_j' = \frac{(\beta + \delta) w_j + 2\nu (z - \zeta)}{h}.$$

Passing through $h = 0$ changes the sign of the denominator, requiring that that of the numerator be preserved; this requires that

$$z \neq \frac{(\beta + \delta) \xi + \nu\zeta}{\beta (\beta + \delta) + \nu}.$$

Similarly,

$$w_i' = \frac{[(\beta + \delta) w_i + 2\nu (z - \zeta)] h - \frac{1}{2} w_i [(\beta + \delta) w_j + 2\nu (z - \zeta)]}{h^2}.$$

Now, though, the numerator must change its sign while h passes through zero, requiring that

$$z = \frac{(\beta + \delta) \xi + \nu\zeta}{\beta (\beta + \delta) + \nu};$$

and incompatible with the requirement from w_j'.

Therefore, not both w_1 and w_2 can be cusps at the same time in system 3.9, violating the definition of a cusp. ■

As $h = 0 \Leftrightarrow \dot{z} = 0$, and as solutions to system 3.9 may be written as the parameterised curves $(w_1(z(t)), w_2(z(t)), z(t))$, points at which $h = 0$ are also points at which tangents do not exist to these solution paths. As these points are not cusps they must represent asymptotes.

To conclude, the small determinant test does not discard inverted inflection points as systems 3.6 and 3.9 do not have any. It does not discard cusps from system 3.9 for the same reason; in the case of system 3.6, the small determinant test is supplemented by a direct test of $g \approx 0$.

There are also other conditions that discard paths from further consideration. Tests for these have also been implemented in the code. Quasi-non-invertible paths, for example, are identified by sign tests on the derivatives when moving across $w_i = -2\xi$. Paths setting $W_i(z) > 0$ for some z are detected by means of Bellman equation 3.2. As this relates $W_i(z)$ and $w_i(z) \equiv W_i'(z)$, and as integration determines $w_i(z)$, $W_i(z)$ is easily calculated.

It may also be determined when a path along which both agents have cornered ($w_i(z) \leq -2\xi, i = 1, 2$) stays in the corner. This calculation implements page 57's Lemma 3.1, which provided a sufficient condition for exactly this outcome. In these cases, the path is then tested against the transversality conditions of Section 3.3.2.

If any of these conditions is met, or if $\bar{z}$, the (finite) upper limit of integration is reached, integration terminates and the next path in the grid is selected. An upper limit of integration of $\bar{z} = 1 \times 10^{17}$ has sufficed to ensure that all paths selected by the grid method will fail at least one of these conditions before reaching $\bar{z}$.[1]

The integration routines used are from the Numerical Algorithms Group (NAG).[2] For initial value problems with high accuracy requirements, the NAG library recommends Adams methods when the system is not stiff.[3] The present code therefore uses the `d02cjc` ordinary differential equation solver, a variable-order, variable-step Adams method. When `d02cjc` fails to make further progress the `d02ejc` ordinary differential equation solver for stiff systems sometimes makes more headway; it uses a variable-order, variable-step backward differentiation formula.[4]

---

[1]The NAG routine `d02cjc` chooses its first step size as a function of $\bar{z} - z$. Increasing the upper limit of integration has caused one or two paths, originally discarded as non-invertible, to become discarded for setting $W_i(z) > 0$, and vice versa. As, in either case, these paths were discarded, this instability is unlikely to affect the equilibrium set.

[2]The implementation code of the NAG routines used is CLSOL05DA, the NAG C Mark 5 library for Sun SPARC Solaris Double Precision operating systems. See `www.nag.co.uk` for more information.

[3]The stiffness ratio of a system is defined as

$$s \equiv \frac{\max_i \mu_i}{\min_i \mu_i} \; i \in \{1, 2\};$$

where μ_i is the real component of the i^{th} eigenvalue of the (linearised) system. An ordinary differential equation system in z is stiff if $\mu_i < 0, i = 1, 2$ and $s >> 1$. A nonlinear system in which s varies is stiff in an interval I when the above hold and $z \in I$ [Itô86, 303.G].

[4]NAG sample code often uses the square root of machine zero as the tolerance; on the

3.5.4 Conditioning

As numerical computation relies upon finite approximations to real numbers, certain operations risk dropping the number of significant digits carried to below acceptable levels. The condition number of a system is a crude approximation to the number of digits lost: when expressed as a power of 10, the exponent reflects the number of significant digits lost in an operation.

Condition numbers may be calculated in a number of different ways, usually in agreement as to order of magnitude. Particularly easy to compute is that based on the L^∞ norm:

Definition 3.32 *For a matrix* $\mathbf{A}$ *with elements* a_{ij} *and inverse* $\mathbf{A}^{-1}$ *with elements* a_{ij}^{-1} *the* L^∞ norm condition number *is*

$$cond_\infty(\mathbf{A}) \equiv \max_{i,j}\{|a_{ij}|\} \times \max_{i,j}\left\{\left|a_{ij}^{-1}\right|\right\}.$$

As computation is performed on a Sun SuperSPARC 1000 with 15 - 16 significant digits, convention calls a path *poorly conditioned* when $cond_\infty > 10^{10}$ [Jud98, §3.5].

The condition numbers of systems 3.6 and 3.9 are

$$cond_\infty(\mathbf{A}) = \frac{\max\left\{\frac{1}{4}w_1^2, \frac{1}{4}w_2^2, g^2, 1\right\}}{|\det(\mathbf{A})|}, w_i \geq -2\xi, i = 1, 2; \tag{3.41}$$

and

$$cond_\infty(\mathbf{A}) = \max\left\{\left(\frac{w_i}{2h}\right)^2, 1\right\}, w_i \leq -2\xi, i = 1, 2; \tag{3.42}$$

respectively. In both cases, a non-invertible $\mathbf{A}$ is sufficient for poor conditioning. Call a path *acceptably poorly conditioned* if it sets $\det(\mathbf{A}) \approx 0$ and if its neighbours are also poorly conditioned. Discard these paths from consideration as candidates by the following argument: it is unlikely that the path has been falsely identified as non-invertible (a sufficient condition for discarding it) due to a round-off error as its neighbours suffer the same fate. It is unclear whether this argument is sound; it has not been encountered in the papers cited here.

Conversely, if a large numerator causes poor conditioning, this may be unacceptable and more careful investigation would be warranted.

3.6 Results

The parameter values usually used in the numerical analyses are displayed in Table 3.1. They are labelled 'multiple' or 'unique' on the basis of whether they give rise to multiple or unique MPE when play is symmetric. Inequality

present hardware this convention implies that `TOL`$= 10^{-8}$.

	β	δ	ν	ξ	ζ
Multiple symmetric EQ	$\frac{1}{10{,}000}$	0	5.4×10^{-7}	1.5	760
Unique symmetric EQ	$\frac{1}{115}$	$\frac{1}{100}$	5.4×10^{-7}	1.5	760

Table 3.1: Sets of parameter values most commonly used

3.38, which described when the paths through a singularity would always be real, holds for neither calibration, allowing the possibility of a more complicated 2-singularity locus. Neither calibration satisfies condition C.4 of Appendix C, the sufficient condition for corner play to be a best response; this rules out solutions in which $x_i(z) = 0 \forall z$ for some i.

In the 'multiple' calibration, inequality 3.39 (inequality 3.38 after the variable transformation) becomes $q < 2$ or $q > 161$. These imply either that $(p-1)^2 < 0$ or that $p^2 - 161p + 1 > 0$. The former yields a contradiction, the latter that $p \lesssim .0062$ or $p \gtrsim 160.9938$ (where the values are each others' reciprocals). Starting from the symmetric $p = -1$, then, it is only possible to move so far in the direction of asymmetry with p increasing before the 2-singularity locus disappears. In this case, by equation 3.28, $w_1 \approx 25,919 w_2$ is the most extreme asymmetry possible. In the present calculations, the boundary of $w_i = -2\xi$ is hit when $w_1 = -3$ and $w_2 \approx -1.04$, well within these limits. The 2-singularity locus is therefore not complicated.

Results of running the code with these parameter values are presented in the following figures. The case of the multiple symmetric equilibria is presented in Figure 3.3 and that of the unique symmetric equilibrium in Figure 3.6. The axes represent the initial conditions $w_1(0)$ and $w_2(0)$; when these are in $(-3\xi, -2\xi)$ they correspond to starting play in the corner, $\mathbf{x}(0) = \mathbf{0}$. The regions that result represent the outcome of the paths starting with these values; they do not indicate the level of z, for example, at which the outcome occurs. Under both calibrations the linear equilibria are unique in the class of linear equilibria, consistent with the result presented in Lockwood [Loc96].

3.6.1 Multiple symmetric equilibria

The results of numerical integration based on the grid of initial conditions are examined first for signs of MPE. Attention is then turned to the possibility of one dimensional MPE by investigating one dimensional loci of potential interest.

Grid of initial conditions

The parameter values generating multiple symmetric equilibria produce Figure 3.3. A consistency check may be performed by comparing results along the symmetric axis to the analytical results displayed in, for example, Figure

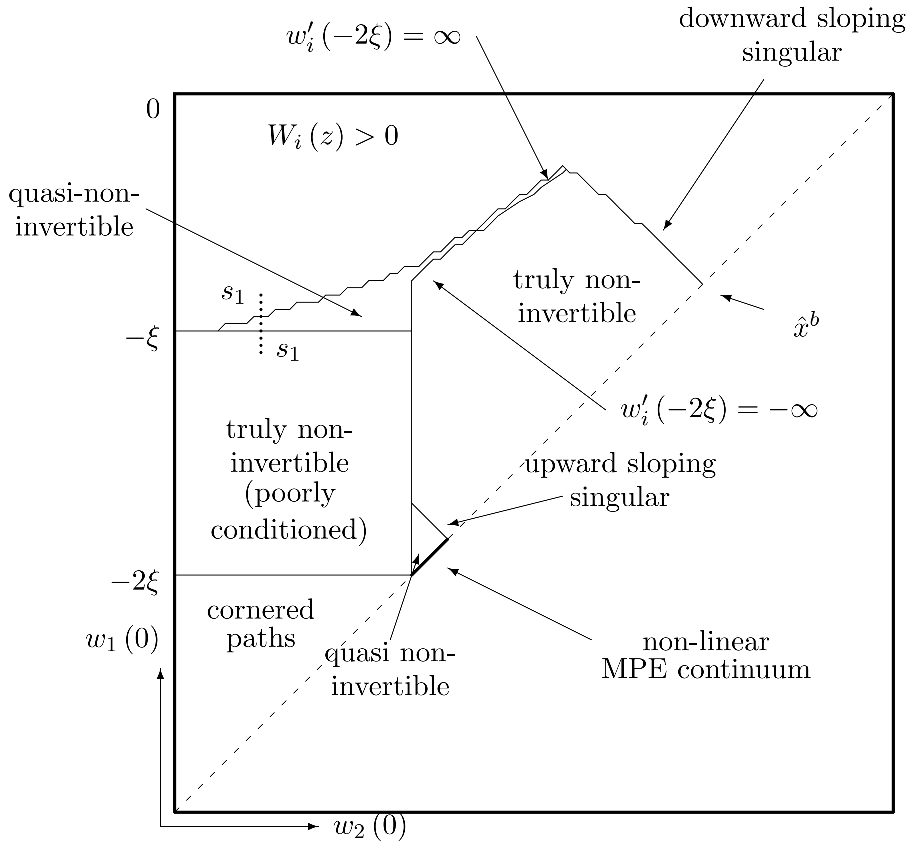

Figure 3.3: Outcome as a function of 100 × 100 initial conditions (multiple symmetric equilibrium)

2.2. They match: paths with $w_i(0) < -2\xi$ never set $x_i > 0$, corresponding to the x^0 analytical paths. Beyond the $w_i(0) = -2\xi$ border there is a region of MPE paths, the $\hat{x}^3$ family. The analytical world follows these by an upward sloping linear path, x^a, before the x^4 and x^5 paths; in Figure 3.3 these latter correspond to the paths in the truly non-invertible region. Analytically, the next boundary is the $\hat{x}^b$ MPE path, after which the unbounded x^6 paths appear; Figure 3.3 possesses these features.

Now asymmetric MPE strategies may be sought.

There is no evidence of regions of asymmetric MPE. The asymmetric neighbours of the continuum of MPE paths might have been expected to be but robustness fails: even asymmetries as small as machine zero lead to quasi-non-invertibility. This occurs as the symmetric strategies that support the MPE obey equations 3.14 after cornering; their neighbours obey equations 3.9.

Given that the grid technique does not reveal asymmetric MPE, they may be sought in two more places. First, there may be regions of MPE too small to be detected by the grid technique. Second, the MPE may be one dimensional or isolated in $(w_1(0), w_2(0))$ space. The following pursues this latter possibility by exploring the boundaries between regions.

One dimensional MPE?

Given the relationship in Figure 2.2 between the 2-singularity there and the linear MPE strategy, paths through the 2-singularity locus are examined here. Because normal integration is not possible near this locus the following method is used:

1. integration off of any point on the 2-singularity locus initially occurs by solving quadratic equation 3.24 for λ. Knowledge of $\mathbf{r}$ then determines the two slopes, allowing initial movement off the locus to be calculated.

2. when the absolute value of $det(\mathbf{A})$ exceeds zero by some tolerance, the NAG integration routines are again used.

Although condition 3.38 does not hold for all p for this calibration, two paths through each singular point are found by this method over the range calculated, as anticipated in the previous discussion.

The results of this procedure are displayed in Figure 3.4. The paths increasing to the left are therefore $\hat{x}^b$ and its asymmetric siblings; those increasing to the right are x^a and its siblings. The two loci of points for which these paths intersect $z = 0$ produce the boundaries in Figure 3.3 noted above.

Integration reveals that none of the asymmetric paths are equilibria. The increasing paths set $W_i(z) > 0$ while the decreasing paths become quasi-non-invertible when they intersect $w_i = -2\xi$. In the former cases, the asymmetric increasing paths seem to approximate x^a as z increases. In the latter cases, paths closer to the far end of the singularity locus actually loop back on themselves, passing through the same point on the singularity locus. Those paths setting $W_i(z) > 0$, beyond the end of the 2-singularity locus, do not cross $w_i = -2\xi$.

There is therefore no evidence of new MPE paths along the singular locus.

Some other boundaries also bear investigation. That between the cornered and the (poorly conditioned) truly non-invertible paths simply divides paths into those with zero emissions initially and those with positive emissions.

The next boundaries to the north (near $w_1(0) = -\xi$) are explored by integrating along a section of initial conditions, $s_1 - s_1$, as displayed in Figure 3.5. The southernmost of these paths becomes non-invertible with w_2 very

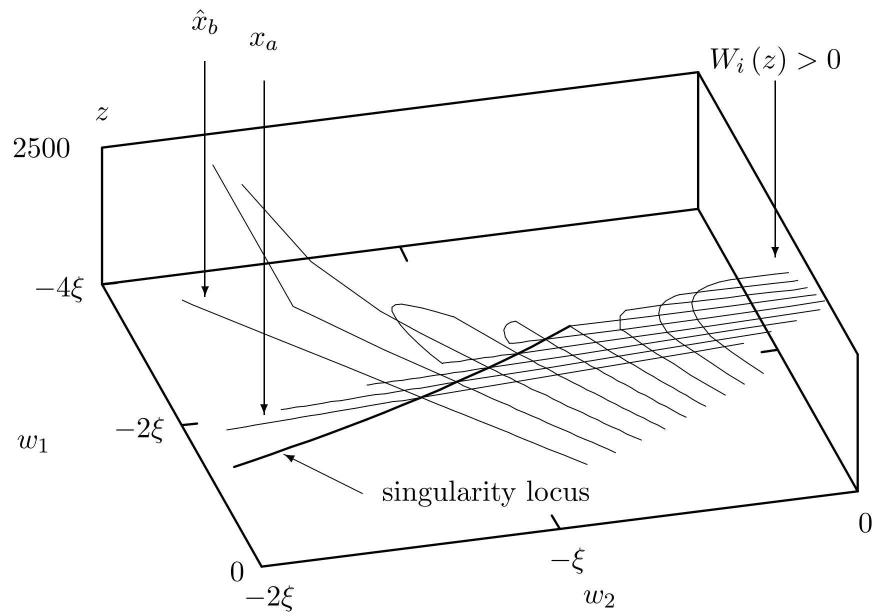

Figure 3.4: Paths through and near the 2-singularity locus

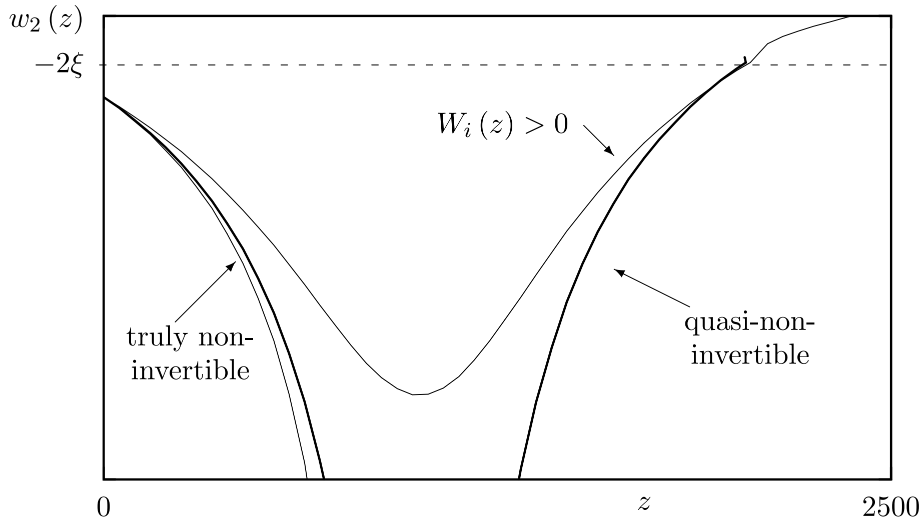

Figure 3.5: Projections of paths through the s_1 section of initial conditions with fixed $w_2(0)$

negative and large.[5] When $w_2 < -2\xi$, the condition for non-invertibility is that $h = 0$, or $w_1 = 2(\beta z - \xi)$, a line in (w_1, z) space. The second path starts at $w_1(0) > -\xi$; this returns from the corner, but becomes quasi-non-invertible in doing so. The third path returns properly from the corner but ultimately sets $W_i > 0$.

This pattern is repeated to the east-north-east, on the other side of $w_2(0) = -2\xi$ (or $x_2(0) = 0$), with a single difference: while still becoming truly non-invertible, the southernmost paths now start in the interior; they do not corner and they remain well conditioned. The quasi-non-invertible paths again fail to leave the corner, while those that set $W_i(z) > 0$ do leave the corner (or never corner if the initial conditions are large enough).

By integrating along paths close to the NW border between the $W_i(z) > 0$ paths and the quasi-non-invertible paths to their south it may be seen that this border sets $w_i'(-2\xi) = \infty$. Similarly, the border to the south, separating quasi- and truly non-invertible paths marks $w_i'(-2\xi) = -\infty$. As none of these boundaries allow the paths through them to be MPE, no evidence is found for asymmetric MPE.

In conclusion, confidence in the belief that asymmetric MPE strategies do not exist is good. None have been found, either by the initial grid search or by more specific searches attuned to one dimensional loci of interest; all of the poorly conditioned paths are acceptably so; there is no evidence of cusps.

3.6.2 Unique symmetric equilibrium

As the results here are simpler than those above, they are presented in more cursory fashion. Again, there is no evidence of new regions of MPE. The symmetric axis of Figure 3.6 is consistent with analytical results under symmetric play. Initially paths corner; now these paths are not just the x^0 family but $\hat{x}^a$ and some x^4 paths. When starting values are large enough for paths not to corner they are the x^4 and x^5 paths. The grid of initial conditions displayed in Figure 3.6 does not extend to sufficiently high initial conditions to demonstrate the $\hat{x}^b$ path and the x^6 family but individual integration with initial conditions up to $w_i(0) = \xi$ do reveal them.

Novel here are the large areas in which even the hardier `d02ejc` integration routine has failed. All individual paths explored are found to approach the non-invertible $h = 0$, accompanied by an exponential decline in one of the $w_i(z)$; these are therefore acceptable examples of poor conditioning. The border through which section s_2 (indicated in Figure 3.6) passes illustrates: to its north, agent 2 corners to approach system 3.9's non-invertible surface; to its south, agent 2 hits the system 3.6 non-invertible surface before cornering.

[5] As the poor conditioning of these paths prevented `d02cjc` from identifying them, `d02ejc` was used.

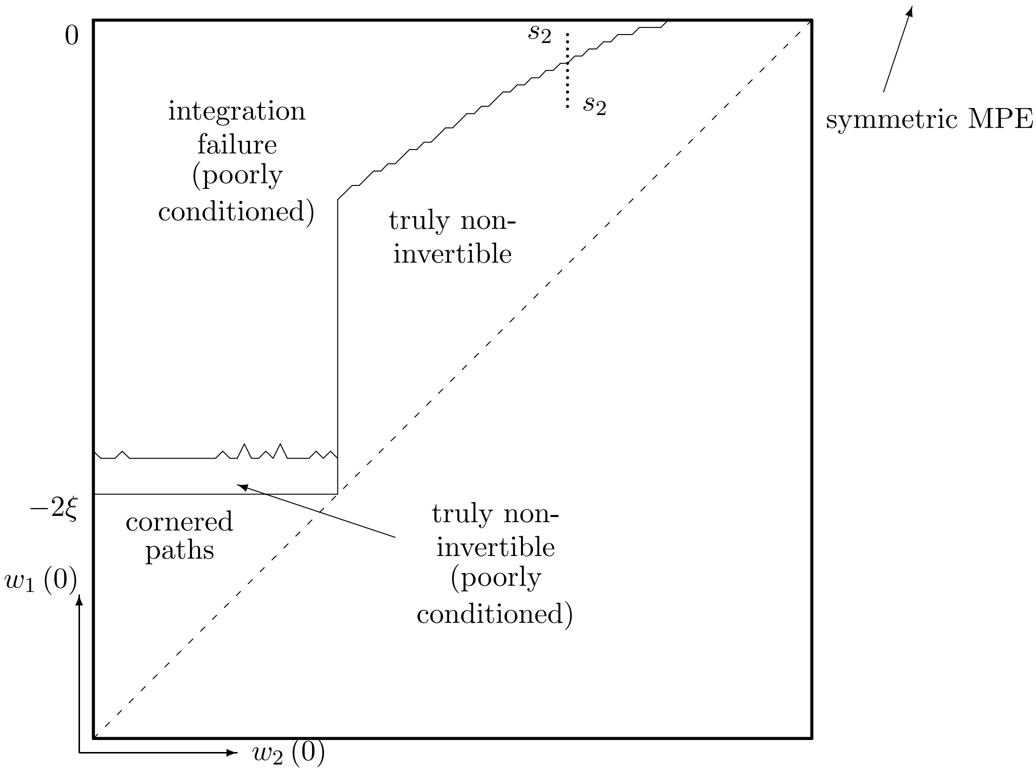

Figure 3.6: Outcome as a function of 100 × 100 initial conditions (unique symmetric equilibria)

Again, there is no sign that cusps have been detected.

3.7 Discussion

This paper suggests that the non-linear equilibria of Chapter 2 are not robust: their neighbouring asymmetric paths do not support MPE. Furthermore, this paper finds no evidence of new asymmetric MPE. This is a weaker statement than a formal proof, which might be constructed by extending the proof technique used in Chapter 2. In that case, all possible families of solutions to equation 3.8 were identified on the ensuing phase diagram, Figure 2.2. Asymmetric play does no more than add additional dimensions: families of paths could be identified in $\Re^3$ and analogous arguments used. For example, those paths to the 'right' of the singularity locus in Figure 3.4, between the paths through the singularity, seem to converge towards x^a; if they did, that would be sufficient for their dismissal.

One of the features of the symmetric system is that it has both one fewer equation and one fewer unknown than does the system that allows asymmetric play. As the equations are not simple linear equations, this counting technique should not account for the non-robustness of the continuum. Rather it seems that the case of symmetric play is simply a special case, exhibiting properties not elsewhere found in the unrestricted system. This special aspect of symmetric play may also give rise to the continuum result found in the model with logarithmic consumption utility presented earlier in Appendix B.

Initially Wirl and Dockner's model of a monopoly supplier and a monopsony demander [WD95] seems to provide a counter-example to this possible link between agent symmetry and multiplicity. In spite of their agents' asymmetry, they too find a continuum of MPE. They do this analytically, first summing the two differential equations describing candidate value functions into a single one to define a new candidate value function. They then follow the approach of Tsutsui and Mino's 1990 paper [TM90]. This second step reduces confidence in their analysis for the reasons outlined in Chapter 2.

The pleasant computational implication of the suspected uniqueness of MPE strategies is that calculation of MPE for more than two agents may be reduced, in the linear-quadratic model, to the solution of coupled Riccati equations. Uniqueness may also have practical consequences for those situations reflected in this model: Pareto improvements cannot be obtained by coordinating on superior Nash equilibria (cf. Radner [Rad98, p.8]).

For games that are not linear-quadratic, though, it is unclear what conclusions may be drawn from the present analysis. In those cases the Riccati equations are not generally of use. It should not be expected that the 2-singularity locus derived here might apply to them. It is to be expected that individual exploration of non-linear-quadratic games remains necessary; the techniques used here should provide much of what is needed.

Finally, while the control bounds considered here are one-sided (≥ 0) the present techniques are independent of this assumption. Analysis could be extended easily to include an upper bound as well. In this case there would be 3^2 scenarios to consider rather than the present 2^2.

Appendix D

Simplifying the discriminant

This appendix contains the proof of Lemma 3.21 (q.v. page 69). That presented a necessary and sufficient condition for the quadratic equation in Theorem 3.13 to have a positive discriminant. The bulk of this appendix is devoted to producing a quadratic expression in r_1 as a function of the parameters $(p, \beta, \delta, \nu, \xi, \zeta)$; the expression for the discriminant then follows.

As the general quadratic equation 3.19 has two roots, let $w_i' = r_i, i = 1, 2$, so that r_i does not need correction by a λ term to produce a slope. Note also that

$$r_3 = g = \frac{2\xi - \beta z}{1 - p - \frac{1}{p}};$$

where the last equality comes from the definition of g and the generating lines of equation 3.28. The (r_1, r_2) terms require more calculation. When $g \neq 0$, system 3.36 (system of differential equations 3.6 at non-invertibility) may be written as

$$\left(r_1 + \frac{1}{p} r_2\right) \left[\begin{array}{c} 1 \\ p \end{array}\right] = 2\left(\beta + \delta\right) \left[\begin{array}{c} \frac{1}{p} \\ p \end{array}\right] + 2\nu\left(z - \zeta\right) \frac{\left(1 - p - \frac{1}{p}\right)}{\left(2\xi - \beta z\right)} \left[\begin{array}{c} 1 \\ 1 \end{array}\right]. \quad \text{(D.1)}$$

Dividing the second by $p \neq 0$ to equate their RHS leads to

$$2\left(\beta + \delta\right)\left(\frac{1}{p} - 1\right) = \left(\frac{1}{p} - 1\right) 2\nu\left(z - \zeta\right) \frac{\left(1 - p - \frac{1}{p}\right)}{\left(2\xi - \beta z\right)};$$

so that $p \neq 1$ implies that

$$\left(\beta + \delta\right) = \nu\left(z - \zeta\right) \frac{\left(1 - p - \frac{1}{p}\right)}{\left(2\xi - \beta z\right)}.$$

When $z \neq \frac{2\xi}{\beta}$

$$z = \frac{2\xi\left(\beta + \delta\right) + \left(1 - p - \frac{1}{p}\right)\nu\zeta}{\beta\left(\beta + \delta\right) + \left(1 - p - \frac{1}{p}\right)\nu};$$

so that

$$g = \frac{(2\xi - \beta\zeta)\,\nu}{\beta\,(\beta+\delta) + \left(1 - p - \frac{1}{p}\right)\nu}.$$

Violation of the assumed $z \neq \frac{2\xi}{\beta}$ causes $g = 0$ and $rank\,(\mathbf{A}\,(\sigma)) = 0$, a violation of the first condition of Theorem 3.13.

Substituting into equation D.1 then produces

$$r_2 = 2\,(\beta+\delta)\,(1+p) - pr_1. \tag{D.2}$$

Differentiate system 3.6 at non-invertibility for

$$\left[\begin{array}{cc} g & \frac{1}{p}g \\ pg & g \end{array}\right]\left[\begin{array}{c} w_1'' \\ w_2'' \end{array}\right] + \left[\begin{array}{cc} g' & \frac{1}{2}w_1' \\ \frac{1}{2}w_2' & g' \end{array}\right]\left[\begin{array}{c} w_1' \\ w_2' \end{array}\right] = (\beta+\delta)\left[\begin{array}{c} w_1' \\ w_2' \end{array}\right] + 2\nu\left[\begin{array}{c} 1 \\ 1 \end{array}\right].$$

Premultiplying by $\left(\begin{array}{cc} c_1 & c_2 \end{array}\right) = \left(\begin{array}{cc} 1 & -\frac{1}{p} \end{array}\right)$ then produces

$$\left[\begin{array}{cc} g' - \frac{1}{p}\frac{1}{2}r_2 & \frac{1}{2}r_1 - \frac{1}{p}g' \end{array}\right]\left[\begin{array}{c} r_1 \\ r_2 \end{array}\right] = (\beta+\delta)\left(r_1 - \frac{1}{p}r_2\right) + 2\nu\left(1 - \frac{1}{p}\right).$$

As $g' = \frac{1}{2}(r_1 + r_2) - \beta$ this simplifies to

$$\begin{aligned} &\tfrac{1}{2}r_1^2 - \tfrac{1}{p}\tfrac{1}{2}r_2^2 + \left(1 - \tfrac{1}{p}\right) r_1 r_2 \\ &= (2\beta+\delta)\,r_1 - \tfrac{1}{p}\,(2\beta+\delta)\,r_2 + 2\nu\left(1 - \tfrac{1}{p}\right). \end{aligned} \tag{D.3}$$

Substitute equation D.2 into equation D.3 for

$$\begin{aligned} & \frac{1}{2}r_1^2 - \frac{1}{p}\frac{1}{2}\left[2\,(\beta+\delta)\,(1+p) - pr_1\right]^2 \\ & + \left(1 - \frac{1}{p}\right) r_1 \left[2\,(\beta+\delta)\,(1+p) - pr_1\right] \\ = \;& (2\beta+\delta)\,r_1 - \frac{1}{p}\,(2\beta+\delta)\left[2\,(\beta+\delta)\,(1+p) - pr_1\right] \\ & + 2\nu\left(1 - \frac{1}{p}\right); \end{aligned}$$

which reduces to the canonical form

$$\begin{aligned} & r_1^2 + \frac{4}{3}\frac{\left[(\beta+\delta)\,(1+p)\left(2 - \frac{1}{p}\right) - (2\beta+\delta)\right]}{(1-p)} r_1 \\ & \frac{4}{3}\frac{\left[(\beta+\delta)\left(\frac{1}{p} + 1\right)(2\beta+\delta) - (\beta+\delta)^2\left(\frac{1}{p} + 2 + p\right) - \nu\left(1 - \frac{1}{p}\right)\right]}{(1-p)} \\ = \;& 0. \end{aligned}$$

The discriminant of this is

$$\frac{16}{9p^2(p-1)^2}\Big\{[\beta+\delta]^2\,p^4 - \big[\beta^2+3\nu+\beta\delta\big]\,p^3 - \big[2\beta\delta+\delta^2-6\nu\big]\,p^2 - \big[\beta^2+3\nu+\beta\delta\big]\,p + [\beta+\delta]^2\Big\}.$$

When $p \neq 1$, a positive discriminant therefore requires that

$$[\beta+\delta]^2\,p^4 - \big[\beta^2+3\nu+\beta\delta\big]\,p^3 - \big[2\,\beta\,\delta+\delta^2-6\,\nu\big]\,p^2 - \big[\beta^2+3\nu+\beta\delta\big]\,p + [\beta+\delta]^2 > 0.$$

Chapter 4

Two asymmetric agents

4.1 Introduction

Using the same model and techniques developed in Chapter 3, this chapter considers a case in which asymmetric agents play asymmetrically. The case is chosen as a technical example, rather than to model a feature of the real world. In it, agents are identical save that one is perfectly patient while the other is not. As before, there are no new non-linear MPE found.

The code is only slightly generalised to increase the parameter space. This chapter is therefore quite short.

Section 4.2 presents the new model, Section 4.3 the results and Section 4.4 concludes.

4.2 The linear-quadratic model with asymmetric agents

Generalise the instantaneous utility functions to

$$u_i\left(x_i, z\right) = -\left(x_i - \xi_i\right)^2 - \nu_i\left(z - \zeta_i\right)^2, i = 1, 2;$$

and use the linear equation of motion 2.3,

$$\dot{z}\left(t\right) = x_1\left(z\right) + x_2\left(z\right) - \beta z\left(t\right).$$

Each agent's Bellman equation is now

$$\delta_i V_i\left(z\right) = \max_{x_i \geq 0} \left\{u_i\left(x_i, z\right) + V_i'\left(z\right)\left(x_1 + x_2 - \beta z\right)\right\}; \qquad (4.1)$$

which has the first order condition

$$x_i^* = \max\left\{0, \xi_i + \frac{1}{2}V_i'\left(z\right)\right\}.$$

The slight generalisation introduced here does not cause the model to violate the Lockwood [Loc96] conditions for unique affine MPE strategies.

Again, there are 2×2 scenarios. When $W\left(\right)$ refers to a candidate strategy, as before, the Bellman equation in x_i^*, differentiated and with $w_i(z) \equiv W_i'(z)$, is

$$\begin{aligned} w_i'(z)\left(x_1^* + x_2^* - \beta z\right) &= \left(\beta + \delta_i - x_1^{*\prime} - x_2^{*\prime}\right) w_i(z) \\ &\quad + 2\nu_i\left(z - \zeta_i\right) + 2x_i^{*\prime}\left(x_i^* - \xi_i\right). \end{aligned}$$

The equations associated with each of the scenarios are as follows.

4.2.1 Both agents interior

When both agents are in the interior, $x_i^* > 0 \forall i = 1,2 \Rightarrow x_i^* = \xi_i + \frac{1}{2}w_i(z), x_i^{*\prime} = \frac{1}{2}w_i'(z) \, \forall i = 1,2$ with Bellman system

$$\begin{bmatrix} g & \frac{1}{2}w_1 \\ \frac{1}{2}w_2 & g \end{bmatrix} \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} (\beta + \delta_1) w_1 + 2\nu_1(z - \zeta_1) \\ (\beta + \delta_2) w_2 + 2\nu_2(z - \zeta_2) \end{bmatrix}; \qquad (4.2)$$

where

$$g \equiv \frac{1}{2}(w_1 + w_2) - \beta z + (\xi_1 + \xi_2);$$

so that $\dot{z} = g(\mathbf{w}, z)$.

The non-invertibility locus is still defined by $g^2 = \frac{1}{4}w_1 w_2$ but the spanning condition is now

$$(\beta + \delta_1) w_1 w_2 - (\beta + \delta_2)(2g) w_2 + 2\nu_1(z - \zeta_1) w_2 - 2\nu_2(z - \zeta_2)(2g) = 0;$$

an algebraic variety of degree two. This reduces to equation 3.29, the intersecting line and plane, in the degenerate case of symmetric agents. Sample vectors involved in non-invertibility and spanning, according to the notation of Definition 3.12 and Theorem 3.13, are then

$$\begin{aligned} \mathbf{q} &= \left(-sign(w_i)\sqrt{\frac{w_1}{w_2}}, 1, 0\right)'; \\ \mathbf{c} &= \left(-sign(w_i)\sqrt{\frac{w_2}{w_1}}, 1, 0\right); \\ \mathbf{r} &= g \times \left(2(\beta + \delta_2), 2\frac{\nu_2}{g}(z - \zeta_2), 1\right)'. \end{aligned}$$

4.2.2 One agent interior, the other cornered

Let agent j be in the interior and agent i be on the corner. Then $x_j^* > 0, x_i^* = 0 \Rightarrow x_j^* = \xi_j + \frac{1}{2}w_j(z), x_j^{*\prime} = \frac{1}{2}w_j'; x_i^{*\prime} = 0$ with differential Bellman equation system

$$\begin{bmatrix} h & 0 \\ \frac{1}{2}w_i & h \end{bmatrix} \begin{bmatrix} w_j' \\ w_i' \end{bmatrix} = \begin{bmatrix} (\beta + \delta_j) w_j + 2\nu_j(z - \zeta_j) \\ (\beta + \delta_i) w_i + 2\nu_i(z - \zeta_i) \end{bmatrix};$$

	β	δ	ν	ξ	ζ
Agent 1	$\frac{1}{10{,}000}$	0	5.4×10^{-7}	1.5	760
Agent 2		$\frac{1}{100}$	5.4×10^{-7}	1.5	760

Table 4.1: Test asymmetric parameter values

where

$$h \equiv \xi_j + \frac{1}{2} w_j - \beta z;$$

so that $\dot{z} = h(\mathbf{w}, z)$.

4.2.3 Both agents cornered

When both agents are cornered $x_i^* = 0 \forall i = 1, 2 \Rightarrow x_i^{*\prime} = 0$ with differential Bellman equation system

$$w_i' = -\frac{(\beta + \delta_i) w_i + 2\nu_i (z - \zeta_i)}{\beta z} \forall i = 1, 2;$$

whose solution is

$$w_i(z) = K_i z^{\frac{\beta+\delta_i}{\beta}} + 2\nu_i \left(\frac{\zeta_i}{\beta + \delta_i} - \frac{z}{\delta_i} \right) = z \left\{ K_i z^{\frac{\delta_i}{\beta}} - \frac{2\nu_i}{\delta_i} \right\} + \frac{2\nu_i \zeta_i}{\beta + \delta_i};$$

where K_i is a constant of integration. As in equation 3.14, the equivalent equation to the above in Chapter 3, $K_i < 0 \forall i = 1, 2 \Rightarrow x_i(z) = 0 \forall i = 1, 2, z \in Z$ and $K_i > 0$ for some $i \in \{1, 2\}$ implies that i will return to the interior.

4.3 Results

The calibration tested here is displayed in Table 4.1. This do not satisfy inequality C.5, which determined when $x_i(z) = 0 \forall z, i \in \{1, 2\}$ might be optimal play.

The results are presented graphically in Figure 4.1. The labelling is slightly simplified here: although many starting values lead to integration failure regions are still labelled according to logic presented below. The 2-singularity locus is not calculated for this model. There is no evidence of regions of MPE.

The multiple symmetric equilibria of Figure 3.3 may be regarded as a limit of the results in Figure 4.1 as $\delta_2 \to 0$. To approximate this, a series of $\delta_2 = \left\{ \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \frac{1}{1000000} \right\}$ has been explored. As expected, these calibrations increasingly resemble Figure 4.1. The poorly conditioned zone setting $W_i(z) > 0$ is immediately replaced by a well conditioned one; the

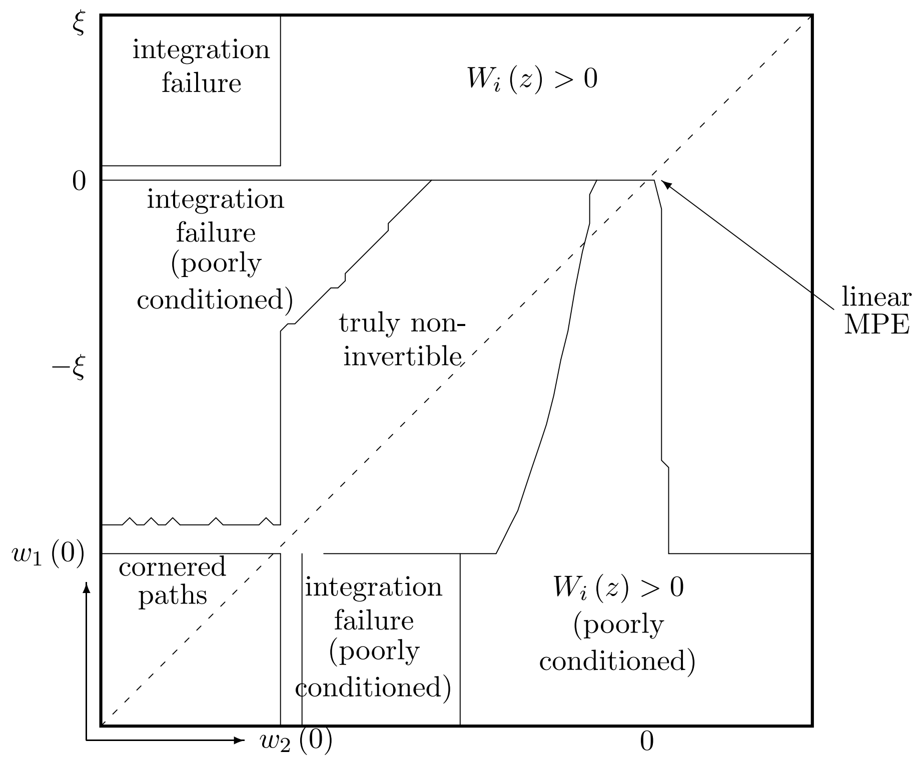

Figure 4.1: Outcome as a function of initial conditions (asymmetric players)

region of failed integration in the north-west disappears; that of poorly conditioned integration failure below it shrinks, its northern boundary becoming a region of quasi-non-invertibility. This sense of continuity allows the guess that these regions of integration failure might be truly non-invertible, by analogy to Figure 3.3. Finally, there is again no evidence of new regions of MPE.

Linear strategies

As Lockwood's conditions apply to system 4.2, it possesses a unique linear MPE. That is now calculated. First define

$$\begin{aligned} V_i &\equiv a_i + b_i z + \frac{1}{2} c_i z^2; \\ v_i &= b_i + c_i z. \end{aligned}$$

By substituting in the interior first order conditions, Bellman equation 4.1 may be rewritten as

$$\begin{aligned} 0 = & \left[(\xi_1 + \xi_2) + \tfrac{(b_1+b_2)}{2} - \tfrac{b_i}{4}\right] b_i - \nu_i \zeta_i^2 - \delta_i a_i \\ & + \left[2\nu_i \zeta_i + (\xi_1 + \xi_2)\, c_i + \tfrac{(b_1+b_2)}{2} c_i + \tfrac{(c_1+c_2)}{2} b_i - \beta b_i - \tfrac{b_i c_i}{2} - \delta_i b_i\right] z \\ & + \left[\tfrac{(c_1+c_2)}{2} c_i - \beta c_i - \tfrac{c_i^2}{4} - \nu_i - \tfrac{\delta_i}{2} c_i\right] z^2, i \neq j \in \{1,2\}\,; \end{aligned} \tag{4.3}$$

the system of coupled algebraic Riccati equations in powers of z. Only those in z and z^2 need solution to determine v_i.

First, solve the nonlinear equations in z^2 for

$$c_i = -2\left\{\frac{1}{2}c_j - \beta - \frac{1}{2}\delta_i \pm \sqrt{\left(\frac{1}{2}c_j - \beta - \frac{1}{2}\delta_i\right)^2 + \nu_i}\right\}, i \neq j \in \{1,2\}\,.$$

The knowledge that the linear MPE path slopes downwards allows this to be refined as

$$c_i = -2\left\{\frac{1}{2}c_j - \beta - \frac{1}{2}\delta_i + \sqrt{\left(\frac{1}{2}c_j - \beta - \frac{1}{2}\delta_i\right)^2 + \nu_i}\right\}, i \neq j \in \{1,2\}\,. \tag{4.4}$$

This system may of two non-linear equations is solved numerically by the NAG routine `c05tbc`, a non-linear equation solver that uses a "modification of the Powell hybrid technique".

Once system 4.4 is solved, the values for $\mathbf{c}$ may be substituted into the coefficients of z in system 4.3. These equations are linear in their unknowns, b_i, so that

$$\begin{aligned} & \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ = & \frac{\begin{bmatrix} \frac{(c_1+c_2)}{2} - (\beta + \delta_2) & -\frac{1}{2}c_1 \\ -\frac{1}{2}c_2 & \frac{(c_1+c_2)}{2} - (\beta + \delta_1) \end{bmatrix} \begin{bmatrix} -2\nu_1\zeta_1 - (\xi_1 + \xi_2)\, c_1 \\ -2\nu_2\zeta_2 - (\xi_1 + \xi_2)\, c_2 \end{bmatrix}}{\frac{(c_1+c_2)}{4}^2 - \frac{(c_1+c_2)(2\beta+\delta_1+\delta_2)}{2} + (\beta + \delta_1)(\beta + \delta_2) - \frac{c_1 c_2}{4}}. \end{aligned}$$

Therefore, for the calibration presented in Table 4.1,

$$\mathbf{v} = \mathbf{b} + \mathbf{c}z \approx \begin{bmatrix} -0.0404 \\ 0.0164 \end{bmatrix} - \begin{bmatrix} 0.0012 \\ 0.0001 \end{bmatrix} z;$$

and $\mathbf{v}(0) \approx (-0.0404, 0.0164)$. This last point is indicated in Figure 4.1 as the linear MPE.

Calculation of $\mathbf{a}$ would allow a preliminary statement to be made about payoffs' relationship to impatience; one expects the patient player to outperform the other. Table 4.1's calibration makes this impossible: a_i only appears in equation 4.3 as a coefficient of δ_i. As $\delta_1 = 0$ here, a_1 is not determined.[1]

4.4 Discussion

There is no evidence of MPE in this model other than the linear MPE. While scrutiny of the numerical results has been more cursory here than in Chapter 3, confidence is still high in this result as nothing seen here has been surprising.

The absence of non-linear equilibria facilitates calculation of equilibria in more realistic calibrations of the model (perhaps along the lines of Nordhaus and Yang's five agents [NY96]) and, consequently, comparative statics.

[1]From the outset, the Bellman equations used here assumed strictly positive discounting. The problem just encountered may illustrate the consequences of perfect patience.

Chapter 5

Transfer functions: non-cooperative cooperation

5.1 Introduction

This chapter explores the thesis' second question, that of non-cooperative cooperation. In doing so, it offers another approach to the general problem of externalities. The two most commonly offered solutions - Pigouvian taxes and Coasian assignment of property rights - depend on the presence of a social planner. Here, by contrast, an approach that does not require a planner is explored by expanding the strategy space available to agents. Now they are allowed to offer transfers contingent upon emissions as reward schemes. This grants agents a second control variable, in contrast to the single control that they have possessed previously.[1]

This interest in transfers follows from two observations. First, transfers have played a role in the political discussion on coordinating greenhouse gas emissions. The Kyoto Protocol agrees in principle that nations may 'trade' emissions in order to meet their emissions reduction targets. This may be interpreted as a conditional transfer mechanism: one country provides a financial transfer to another in return for an agreement on emissions from the latter. The permitted means under Kyoto for trading emissions have yet to be resolved, and are a matter of dispute. Nevertheless, some companies, typically electrical utilities, have made arrangements with foreign governments to, for example, reforest to increase a carbon dioxide sink.

Indeed, transfer-like activity is the stuff of international diplomacy. While much of it will involve in kind reciprocation there are also examples of monetary transfers. US aid to Israel and Egypt increased substantially after President Carter's Camp David accords; its aid to Yemen was withdrawn after the latter's Security Council vote against the use of force to reverse Iraq's 1990 invasion of Kuwait.

[1]Some label games with these larger strategy spaces 'linked games' (q.v. [Car97]).

The second motivation for the consideration of transfers is related to this first: voluntary transfers offer the possibility of Pareto improvement without a social planner. They may therefore be possible in a world such as the present, where there is no international social planner.

To analyse transfers, this chapter specifies transfer strategies as functions. To understand why, consider the usual case, in which agents select specific levels of play. Here, as the Nash equilibrium assumes other agents' play to be fixed, voluntary transfers incur a cost without offering the possibility of altering the others' play. Unless the game is specified in such a way as to yield other benefits to these transfers, agents will not make them in equilibrium. Consideration of transfers therefore requires a solution concept that allows agents to anticipate the possible response of others to their offer of transfers.

Such a solution concept has been provided in Klemperer and Meyer's paper on functional Nash equilibria [KM89]. Their motivating example was an oligopoly problem with demand shocks that occur after firms choose their strategies. Firms therefore select supply functions to respond flexibly to the uncertainty. This does not require any sort of commitment technology as the functions are chosen to so that any outcome is *ex post* optimal. The uncertainty, then, serves at least three roles. First, it serves to motivate the selection of functions. Second, it can assist (as explained in Section 1.3.3) in refining the set of functional equilibria. Third, it may model more accurately the problem at hand.

Note that voluntary transfers are not entirely non-cooperative. Unless the transfer is a lump sum (in which case it is either not a function or is a degenerate one and does not affect incentives at the margin), there is a question of how the transfer takes place: does its recipient play first and hope that the transfer is then given or does the donor make the transfer and hope that it is understood as a function rather than a lump sum? As transfer-like behaviour does take place in the realm of international affairs there is clearly a phenomenon to be explained.

This, of course, is no different than the common game theorist's question of how cooperation arises in non-cooperative games. The usual approach to this is, of course, to look for non-Markov punishment strategies in repeated games. This chapter, however, presents a one-shot model for the sake of tractability. It therefore solves the cooperation problem by assumption: agents offer functions; there is no enforcement problem.

Section 5.2 presents the functional Nash equilibrium concept and related terminology. Section 5.3 explores FNE in a simple model without transfers, presenting an existence and a non-existence result. This and Section 5.4, which develops the Pareto frontier, form baselines for comparison for Section 5.5, which extends the initial simple model to include transfers. Section 5.6 then concludes.

5.2 Functional Nash equilibria

In the class of games examined below a state variable is defined as follows:

$$z \equiv x(z) + y(z) + \varepsilon; \tag{5.1}$$

where agent 1 chooses the function $x(z)$, agent 2 chooses $y(z)$ and ε is a random variable with support $[-\bar{\varepsilon}, \bar{\varepsilon}]$. Let $f(\varepsilon)$ be the density function of ε and $E[\varepsilon] = 0$.[2]

Akin to previous chapters, $z(\varepsilon)$ is a function whose domain is the support of ε. The reasons for this are as follows. First, the problem is otherwise unintelligible: if an ε leads to no z, no $x(z)$ or $y(z)$ can be derived either. Second, the mapping must be a function as, if there were multiple z's for some realisations of ε, a mechanism for picking one of the z's would be required to allow play. Rather than add such a mechanism, strategy pairs giving rise to such a situation are declared inadmissible. This is an *ad hoc* treatment, but perhaps no more so than the alternatives.[3] Furthermore, it is consistent with the standard definition of admissibility in differential games.

As previously, strategies must also be defined over the whole domain, $z \in \Re$. Were they not, agents would be unable to form complete conjectures of others' play, and therefore not be able to select their own best responses. Therefore:

Definition 5.1 *A pair of strategy functions* $(x(z), y(z))$ *is* admissible *if* $x(z)$ *and* $y(z)$ *are defined over* $z \in \Re$ *and if the mapping that they, through equation 5.1, induce between* ε *and* z *is a function with domain* $[-\bar{\varepsilon}, \bar{\varepsilon}]$.

This admissibility requirement is weaker than that used in the differential game as continuity is not required of the state variable.

This definition yields the following restriction:

Lemma 5.2 *An admissible function pair* $(x(z), y(z))$ *produces, through equation 5.1, an injective* $z(\varepsilon)$.

Proof. Argue by contradiction. Consider two values of ε, say ε_1 and ε_2, producing the same z. The left hand side and first two right hand side terms of equation 5.1 are the same for both ε_1 and ε_2 but the last term is different. This cannot be. ■

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[2] The restriction on the first moment plays no role beyond determining expected utilities.

[3] Klemperer and Meyer address this problem by assuming that, "if a market-clearing price does not exist, or is not unique, then firms earn zero profits... [T]his assumption ensures that such an outcome will not arise in equilibrium, but the assumption is not an important constraint on firms' behavior" [KM89, p.1250].

Define expectations over realisations of ε. Therefore, an admissible strategy pair $(x(z(\varepsilon)), y(z(\varepsilon)))$ yields the pair of expected payoffs

$$(E[u(x(z(\varepsilon)))|y(z(\varepsilon))], E[v(y(z(\varepsilon)))|x(z(\varepsilon))]);$$

where $u(x(\varepsilon))$ and $v(y(\varepsilon))$ are the objective functions of agents 1 and 2. Now, for the two agent game, the equilibrium concept is defined as follows:

Definition 5.3 *If:*

1. $(x^*(z(\varepsilon)), y^*(z(\varepsilon)))$ *is admissible; and*
2. $E[u(x^*(z(\varepsilon)))|y^*(z(\varepsilon))] \geq E[u(x(z(\varepsilon)))|y^*(z(\varepsilon))]$; *and*
3. $E[v(y^*(z(\varepsilon)))|x^*(z(\varepsilon))] \geq E[v(y(z(\varepsilon)))|x^*(z(\varepsilon))]$ *for all admissible* $(x(z(\varepsilon)), y^*(z(\varepsilon)))$ *and* $(x^*(z(\varepsilon)), y(z(\varepsilon)))$

then $(x^*(z(\varepsilon)), y^*(z(\varepsilon)))$ *forms a* functional Nash equilibrium (FNE).

5.3 FNE without transfers

Two models are presented here, producing a non-existence and an existence result, respectively. In both cases, the present approach to agents' optimisation problems somewhat resembles that of Groves and Ledyard's approach to the optimal allocation of public goods problem [GL77]. As in that paper, agents here imagine themselves to be choosing the state variable directly as they consider other agents' choices to be fixed. Unlike the analysis in that paper, though, the fixed choices of the other agents are functions rather than merely points. In contrast to the 'competitive' assumption of Groves and Ledyard, then, agents here are allowed to behave strategically. Another key difference between the current analysis and that of Grove and Ledyard is that there is no social planner here to design tax and allocation rules. The optimal allocation of public goods is therefore not expected in equilibrium.

5.3.1 Non-existence: payoffs linear in control

Non-existence results are presented here. They arise from problems in which an agent wishes its control to be constant, but is unable to achieve this due to the uncertainty. First consider a simple case:

Problem 5.4 (linear) *Agents 1 and 2 have objective functions*

$$\begin{aligned} u &= -x-(z-\delta)^2; \text{ and} \\ v &= -y-(z+\delta)^2; \end{aligned}$$

respectively, where δ *is a parameter. Agent 1 solves*

$$\max_{x(z)} E\left[-x-(z-\delta)^2\right];\ z(\varepsilon)=x(z)+y^*(z)+\varepsilon$$

where $y^*(z)$ *is some fixed play by agent 2. Agent 2 solves*

$$\max_{y(z)} E\left[-y-(z+\delta)^2\right];\ z(\varepsilon)=x^*(z)+y(z)+\varepsilon$$

where $x^*(z)$ *is some fixed play by agent 1.*

Theorem 5.5 *There is no FNE in Problem 5.4.*

Proof. Substitute the constraint into agent 1's problem for

$$\max_{z(\varepsilon)} E\left[y^*(z)-z-(z-\delta)^2+\varepsilon\right].$$

The maximising variable is now written as $z(\varepsilon)$ as, for fixed $y^*(z)$, selecting $x(z)$ and $z(\varepsilon)$ are equivalent by equation 5.1. The problem may now be written as

$$\max_{z}\left[y^*(z)-z-(z-\delta)^2\right]; \tag{5.2}$$

by first moving the uncontrollable ε outside of the maximand and noting that the expectations operator may be dropped in the absence of stochastic terms.

As the problem is now deterministic it is solved by some constant $z(\varepsilon)$. Let this constant be ζ so that, by equation 5.1,

$$\zeta=x(\zeta)+y(\zeta)+\varepsilon;$$

a contradiction. As agent 1 has no optimal strategy, there can be no FNE. ■

This result generalises to other problems requiring a constant maximum:

Theorem 5.6 *Suppose that agent 1's problem may be written as*

$$\max_{z(\varepsilon)} E\left[u\left(y^*(z),z,\delta\right)+h(\varepsilon)\right];$$

when its choice variable, x*, is replaced by the constraint*

$$z=f\left(x(z),y^*(z),\varepsilon\right).$$

Let the constant ζ *solve this problem and implicitly define* x^* *according to*

$$\zeta=f\left(x^*(z(\varepsilon)),y^*(z(\varepsilon)),\varepsilon\right); \tag{5.3}$$

where $y^*(z(\varepsilon))$ *is some fixed play by agent 2. For* ζ *not to exist, it is sufficient that* $f_\varepsilon(x^*,y^*,\varepsilon)\neq 0$ *for some realisation of* ε*.*

Proof. As $x^*(\zeta)$ and $y^*(\zeta)$ are constants, a constant ζ, by equation 5.3, that $f_\varepsilon=0\forall\varepsilon$. ■

5.3.2 Existence: payoffs quadratic in control

The following problem resembles that analysed in previous chapters of this thesis. Previously, the instantaneous objective function was quadratic in control and state, as here. Then the equation of motion, whose role equation 5.1 plays, was also linear. This one-shot problem may therefore be loosely interpreted as a limit case of the previous differential games when agents are perfectly patient and z does not decay.

Problem 5.7 (quadratic) *Agents 1 and 2 have objective functions*

$$u = -x^2 - (z-\delta)^2\,; \tag{5.4}$$
$$v = -y^2 - (z+\delta)^2\,; \tag{5.5}$$

respectively, where δ is a parameter. Agent 1 solves

$$\max_{x(z)} E\left[-x^2 - (z-\delta)^2\right];\ z(\varepsilon) = x(z) + y^*(z) + \varepsilon$$

where $y^*(z)$ is some fixed play by agent 2. Agent 2 solves

$$\max_{y(z)} E\left[-y^2 - (z+\delta)^2\right];\ z(\varepsilon) = x^*(z) + y(z) + \varepsilon$$

where $x^*(z)$ is some fixed play by agent 1.

Substituting the constraint into agent 1's problem produces

$$\max_{z(\varepsilon)} E\left[-2z^2 + 2y^*z + 2\varepsilon z - (y^*)^2 - 2\varepsilon y^* - \varepsilon^2 + 2\delta z - \delta^2\right]. \tag{5.6}$$

The multiplicative terms in ε prevent this problem being rewritten as in Theorem 5.5 (attempts to use the constraint to remove the ε terms merely re-introduce the $x(z)$ terms). Problem 5.7 may therefore have a solution. The multiplicative terms also prevent removal of the expectations operator as was done in equation 5.2, above. The problem therefore becomes

$$\max_{z(\varepsilon)} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \left[-2z^2 + 2y^*z + 2\varepsilon z - (y^*)^2 - 2\varepsilon y^* - \varepsilon^2 + 2\delta z - \delta^2\right] f(\varepsilon)\, d\varepsilon. \tag{5.7}$$

Again, regard agent 1 as controlling $z(\varepsilon)$ when y^* is fixed. A necessary condition for a maximum may thus be found by differentiating with respect to z:

$$\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \left[-4z + 2y^{*\prime}z + 2y^* + 2\varepsilon - 2y^*y^{*\prime} - 2\varepsilon y^{*\prime} + 2\delta\right] f(\varepsilon)\, d\varepsilon = 0. \tag{5.8}$$

The injective relationship demonstrated in lemma 5.2 means that agent 1's ability to condition its play on z is equivalent to an ability to condition play on ε. Expected utility maximisation and pointwise maximisation by realisations of ε are therefore equivalent. Necessary condition 5.8 must thus hold for all ε as there is otherwise an $\hat{\varepsilon} \in [-\bar{\varepsilon}, \bar{\varepsilon}]$ such that $z(\hat{\varepsilon})$ is not a stationary point. Therefore, as an extremum requires that the contents of the square bracketed term in equation 5.8 equal zero for all ε:

$$\begin{aligned} y^{*\prime}[z - y^* - \varepsilon] + y^* - 2z + \varepsilon + \delta &= 0; \text{ or} \\ y^{*\prime} &= 1 + \frac{z - \delta}{x^*}. \end{aligned} \tag{5.9}$$

Similarly, the first order necessary condition for agent 2 in Problem 5.7 is

$$x^{*\prime} = 1 + \frac{z + \delta}{y^*}. \tag{5.10}$$

Having implicitly assumed that optimal strategies are elements of $\mathcal{C}^1$, equations 5.9 and 5.10 make them elements of $\mathcal{C}^\infty$ as well. These equations may not possess closed form solutions. While numerical methods could be used to solve them, this section instead examines various special cases, in the hope of finding more tractable first order conditions. In particular, after deriving second order conditions, linear solutions and non-linear solution in symmetric cases (in which $\delta = 0$) are explored.

Second order conditions

Equations 5.9 and 5.10 provide necessary conditions for extrema rather than maxima specifically. Now consider sufficient conditions for maxima. Differentiating the square bracketed term in equation 5.8 with respect to z a second time produces the requirement that

$$\begin{aligned} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \left[-2 + 2y^{*\prime} + (z - y^* - \varepsilon)\, y^{*\prime\prime} - \left(y^{*\prime}\right)^2\right] f(\varepsilon)\, d\varepsilon &< 0; \text{ or} \\ \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \left[-2 + 2y^{*\prime} + x^* y^{*\prime\prime} - \left(y^{*\prime}\right)^2\right] f(\varepsilon)\, d\varepsilon &< 0. \end{aligned}$$

As $z(\varepsilon)$ must maximise for any realisation of $\varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]$, a necessary condition for a maximum is that

$$-2 + 2y^{*\prime} + x^* y^{*\prime\prime} - \left(y^{*\prime}\right)^2 \leq 0. \tag{5.11}$$

For agent 2, the equivalent expression to equation 5.6 is

$$\max_{z(\varepsilon)} E\left[-2z^2 + 2x^*z + 2\varepsilon z - (x^*)^2 - 2\varepsilon x^* - \varepsilon^2 - 2\delta z - \delta\right].$$

The requirement that its second derivative with respect to z be negative produces

$$-2 + 2x^{*\prime} + y^* x^{*\prime\prime} - \left(x^{*\prime}\right)^2 \leq 0. \tag{5.12}$$

To simplify equations 5.11 and 5.12, differentiate equations 5.9 and 5.10 for

$$\begin{aligned} y^{*\prime\prime} &= \frac{1}{x^*} - \frac{z-\delta}{(x^*)^2} x^{*\prime}; \\ x^{*\prime\prime} &= \frac{1}{y^*} - \frac{z+\delta}{(y^*)^2} y^{*\prime}. \end{aligned}$$

These allow equations 5.11 and 5.12 to be written as

$$\frac{z-\delta}{x^*}\left\{1+\frac{z-\delta}{x^*}+\frac{z+\delta}{y^*}\right\} \geq 0; \text{ and} \tag{5.13}$$

$$\frac{z+\delta}{y^*}\left\{1+\frac{z-\delta}{x^*}+\frac{z+\delta}{y^*}\right\} \geq 0. \tag{5.14}$$

It is therefore necessary for a maximum that, for all $z(\varepsilon)$ with $\varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]$, $\frac{z-\delta}{x}$, $\frac{z+\delta}{y}$ and $\left\{1+\frac{z-\delta}{x}+\frac{z+\delta}{y}\right\}$ have the same sign. By defining

$$\begin{aligned} \xi &\equiv \frac{z-\delta}{x}; \text{ and} \\ \eta &\equiv \frac{z+\delta}{y}; \end{aligned}$$

this can be expressed in (ξ, η) space as requiring either that $\xi, \eta \geq 0$ or that $\xi, \eta \leq 0$ with $\xi + \eta \leq -1$. These two conditions define disjoint regions in (ξ, η) space.

Lemma 5.8 *It must be that* $sign\,[x^*(z)] = sign\,[y^*(z)]\ \forall\ z \notin (-\delta, \delta)$ *and that* $sign\,[x^*(z)] \neq sign\,[y^*(z)]\,\forall\ z \in (-\delta, \delta)$.

Proof. This follows from the numerators in first terms of equations 5.13 and 5.14. ∎

Therefore agents 1 and 2 'pull in opposite directions' when $z \in (-\delta, \delta)$ and in the same direction otherwise.

As equations 5.9 and 5.10 may not have closed form solutions, the linear and symmetric special cases mentioned above are now examined.

Linear solutions

Lemma 5.9 *Problem 5.7 has only two linear solutions. These correspond to*

$$\begin{aligned} x^* &= (z-\delta)\,a; \\ y^* &= (z+\delta)\,a; \end{aligned}$$

where a is a constant implicitly defined by

$$a^2 - a - 1 = 0. \tag{5.15}$$

Denote the roots of equation 5.15 by $a_+ > 0 > a_-$. There is thus a form of symmetry between the agents' strategies: x^* is the function of $z - \delta$ that y^* is of $z + \delta$.

Proof. The general formulation for linear strategies of z is

$$\begin{aligned} x &= az + b; \text{ and} \\ y &= cz + d; \end{aligned}$$

where a, b, c and d are real constants. As any extremum of Problem 5.7 must satisfy equations 5.9 and 5.10, any linear solution must satisfy

$$\begin{aligned} a &= 1 + \frac{z+\delta}{cz+d}; \\ c &= 1 + \frac{z-\delta}{az+b}. \end{aligned}$$

Rearrangement produces

$$\begin{aligned} (ac - c - 1)\, z &= d\,(1-a) + \delta; \\ (ac - a - 1)\, z &= b\,(1-c) - \delta. \end{aligned}$$

As these must be satisfied for all z, $c = a$ and $a^2 - a - 1 = 0$. Furthermore, $b = -d = \frac{\delta}{1-a}$, so that

$$\begin{aligned} x &= az + \frac{\delta}{1-a} = \left(z + \frac{\delta}{a - a^2}\right) a = (z+\delta)\, a; \text{ and} \\ y &= (z - \delta)\, a. \end{aligned}$$

As both roots of a satisfy the second order conditions of equations 5.13 and 5.14, these are maxima. ■

Not all solutions to the differential equations 5.9 and 5.10 are solutions to the original maximisation Problem 5.7. The linear solutions are, though:

Theorem 5.10 *The only linear strategies to support a FNE in Problem 5.7 when strategies are restricted to be linear are the two pairs of symmetric strategies identified in Lemma 5.9.*

Proof. Substituting the expression for y^* from Lemma 5.9 into the maximand in agent 1's problem (equation 5.6) produces

$$\max_{z(\varepsilon)} E\left[\left(2a - 2 - a^2\right) z^2 + 2\left(\left(a\delta + \varepsilon\right)(1-a) + \delta\right) z - a^2\delta^2 - 2a\delta\varepsilon - \varepsilon^2 - \delta\right].$$

By Lemma 5.9, $x^* = (z - \delta)\, a$ is an extremum of the maximand. If the maximand is concave in z, then x^* is a maximum given y^* and the result holds. Concavity requires that the coefficient of z^2 be negative. When a is defined as in Lemma 5.9 the coefficient is $-\frac{1}{2}\left\{5 \pm \sqrt{5}\right\} < 0$.

There are no other linear candidates as Lemma 5.9 identified all linear candidate solutions to Problem 5.7. ■

That there are two linear solutions is somewhat troubling as Chapter 2 had only found one linear solution to its differential game. This likely reflects the loose relationship between the previous differential games and the present model.

Therefore:

Theorem 5.11 *The expected payoffs to the linear equilibria in quadratic Problem 5.7 are*

$$E[u] = E[v] = -\left(1+a^2\right)\left(\delta^2 + \frac{1}{5}E\left[\varepsilon^2\right]\right).$$

Proof. Substitution of the linear strategies into the objective functions produces

$$\begin{aligned} E[u] &= -\left(1+a^2\right)\left(\delta^2 - 2\delta E[z] + E\left[z^2\right]\right); \text{ and} \\ E[v] &= -\left(1+a^2\right)\left(\delta^2 + 2\delta E[z] + E\left[z^2\right]\right). \end{aligned}$$

Equation 5.1, which defined the state variable, and the linear strategies allow calculation of $z = \frac{\varepsilon}{1-2a}$. Therefore $E[z] = 0$ and, by equation 5.15, $E\left[z^2\right] = \frac{1}{5}E\left[\varepsilon^2\right]$. The result follows. ■

Although the downward sloping linear path yields a higher payoff than does the upward sloping one, the linear solutions are similar in the following respect. The objective functions are such that both agents would like to set their control to zero and would like $z = 0$. Their strategies are therefore chosen to offset the consequences of a realisation of $\varepsilon \neq 0$. Both strategies produce the same variance for z, a variance less than that of ε.

Symmetric agents and play

Two special cases of symmetric play are now considered. In the first, Problem 5.7 is restricted by setting $\delta = 0$ and requiring that $x(z) = y(z)$, so that agents play symmetrically. This second restriction is removed in the next section.

In the case of symmetric play, though, first order conditions 5.9 and 5.10 reduce to

$$x^{*\prime} = 1 + \frac{z}{x^*}. \tag{5.16}$$

Lemma 5.9 provides the linear solutions to equation 5.16 as special cases; these are plotted in Figure 5.1. Representative non-linear solutions are drawn by noting that $x = -z$ defines the horizontal isocline ($x^{*\prime} = 0$) while $x^* = 0$ defines the vertical ($x^{*\prime} = \pm\infty$).

Now ask which solutions to equation 5.16 support FNE. As the linear solutions are addressed by Theorem 5.10, they are FNE.

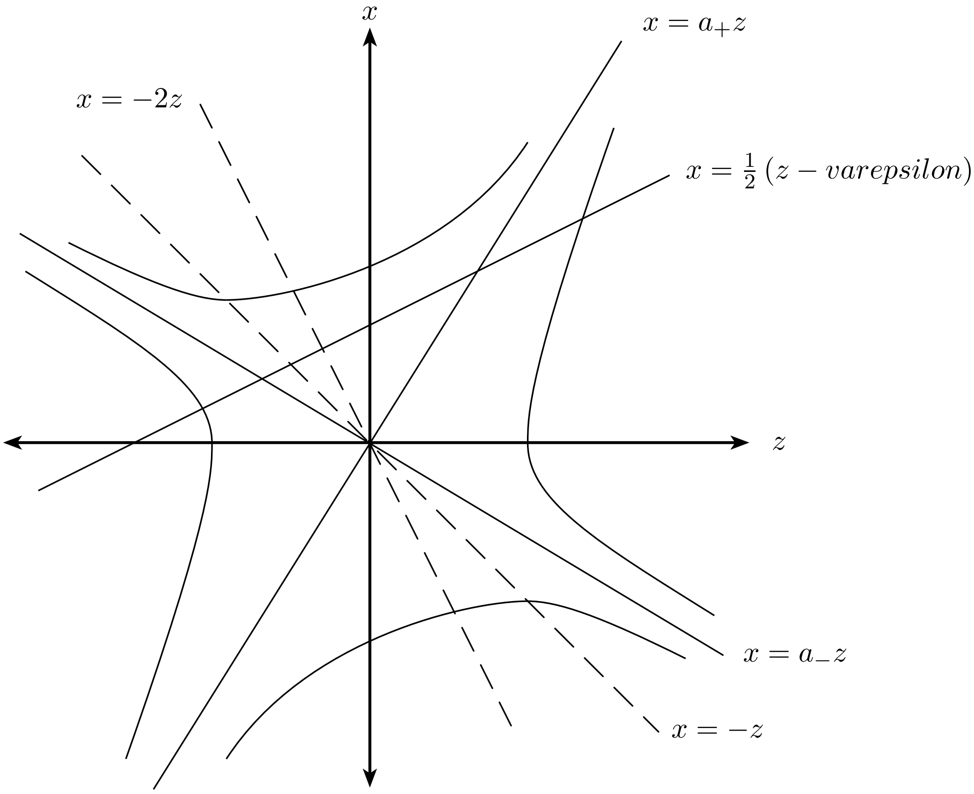

Figure 5.1: Solutions to equation 5.16

Consider now non-linear solutions to equation 5.16. One requirement is that candidate solutions be maxima. As $\delta = 0$ and $x^* = y^*$ in the symmetric cases considered here the second order condition 5.11 becomes

$$-2\left[\left(1 - x^{*\prime}\right)^2 - x^* x^{*\prime\prime} + 1\right] \leq 0. \tag{5.17}$$

As linear solutions set $x^{*\prime\prime} = 0$ it follows that any linear solution to equation 5.16 satisfies the second order condition for a maximum. A simpler condition than that in equation 5.17 may be found by setting $\delta = 0$ and $x = y$ in equation 5.13's version of the second order condition:

$$(2z + x^*)\, z \geq 0. \tag{5.18}$$

The $z = 0$ and $x = -2z$ lines therefore divide (z, x) space into four regions, two satisfying the second order conditions and two violating them.

Turn now to the two sorts of non-linear solutions to equation 5.16 indicated in Figure 5.1.

Lemma 5.12 *The symmetric non-linear solutions to equation 5.16 that cross $x = 0$ are not admissible strategy functions.*

Proof. These $x(z) = y(z)$ are neither defined over all $z \in \Re$ nor are they functions. Such strategy pairs therefore violate the definition of admissible function pairs. ∎

Lemma 5.13 *The symmetric non-linear solutions to equation 5.16 that cross $z = 0$ are not admissible strategy functions.*

Proof. Under symmetry, conditions 5.13 and 5.14 reduce to

$$\frac{z}{x^*}\left(\frac{2z + x^*}{x^*}\right) \geq 0;$$

or $(2z + x^*)\, z \geq 0$. Solutions crossing the $z = 0$ locus therefore violate these conditions in doing so. ∎

The previous two lemmata imply:

Theorem 5.14 *There are no non-linear symmetric strategy pairs that support a FNE in Problem 5.7.*

The proof follows from recognising that no non-linear candidates remain.

The preceding results may be generalised slightly. If $\delta \neq 0$ but both agents had the identical Euler conditions

$$x^{*\prime} = 1 + \frac{z + \delta}{x^*};$$

then similar results to those above would follow. There are still two linear solutions conforming to $x^* = az + b$ but, while a is still $\frac{1}{2}\left\{1 \pm \sqrt{5}\right\} z$, $b = \frac{\delta}{a-1}$. This simply translates the diagram in Figure 5.1.

Symmetric agents and asymmetric play

Continue to restrict Problem 5.7 by setting $\delta = 0$, so that agents are symmetric, but now allow $x(z) \neq y(z)$, so that they may still play asymmetrically.[4] The change of variables

$$\begin{aligned} \xi(z) &\equiv \tfrac{z}{x}; \\ \eta(z) &\equiv \tfrac{z}{y}; \end{aligned} \tag{5.19}$$

then allow

$$\begin{aligned} x' &= \frac{1 - \xi' x}{\xi}; \\ y' &= \frac{1 - \eta' y}{\eta}; \end{aligned}$$

[4] Klemperer and Meyer [KM89, Proposition 3] have a result demonstrating the non-existence of asymmetric equilibria. As their proof relies on the behaviour of their differential equation system and on their assumptions about behaviour when firm profits are negative, the result does not apply directly to the present situation.

so that the first order conditions in equations 5.9 and 5.10 may be written as

$$\begin{aligned} \eta' &= \tfrac{\eta(1-\eta-\xi\eta)}{z}; \\ \xi' &= \tfrac{\xi(1-\xi-\xi\eta)}{z}. \end{aligned} \tag{5.20}$$

By the usual technique this produces

$$\frac{d\eta}{d\xi} = \frac{\eta\,(1-\eta-\xi\eta)}{\xi\,(1-\xi-\xi\eta)}. \tag{5.21}$$

One solution to this is $\xi = \eta$, which becomes $x\,(z) = y\,(z)$ when the change of variables is reversed. This case of symmetric play has already been considered.

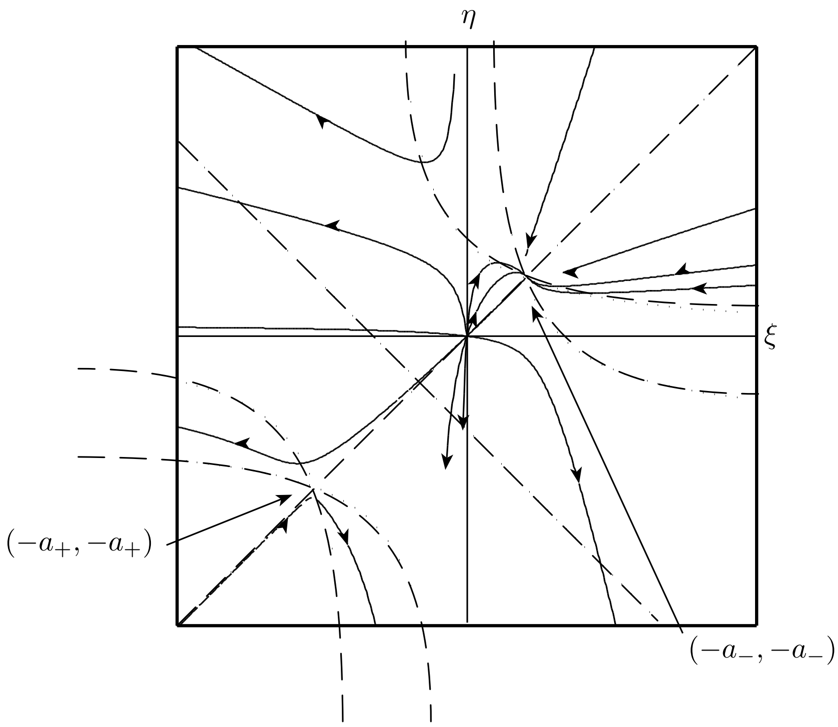

Figure 5.2: Phase diagram in (ξ, η) space

Equation 5.21 allows a phase diagram to be plotted in (ξ, η) space. This is done in Figure 5.2. The flowlines indicate five stationary points at which $\xi' = \eta' = 0$:

$$\mathcal{S} \equiv \{(\xi,\eta) \,|\, (\xi,\eta) \in \{(0,0)\,,(0,1)\,,(1,0)\,,(-a_-,-a_-)\,,(-a_+,-a_+)\}\}\,.$$

The curved dashed lines represent the horizontal and vertical isoclines, those points at which $\frac{d\eta}{d\xi} = 0$ or $\frac{d\eta}{d\xi} = \pm\infty$, respectively.

The downward sloping diagonal dashed line is $\xi + \eta = -1$, one of the boundaries for the second order conditions of equations 5.13 and 5.14. These conditions are satisfied in the first quadrant and in the third below the diagonal line.

The linear solutions found by Lemma 5.9 are now the points $(\xi, \eta) = (-a, -a)$, where $a = \{a_-, a_+\}$. These lie at the two intersections of the horizontal and vertical isoclines; they may be seen to satisfy the second order conditions. By equation 5.21, asymmetric paths through them must satisfy either $\frac{d\eta}{d\xi} = \pm 1$. To understand why, note that, as these points are approached, $\xi \approx \eta$ and

$$\frac{d\eta}{d\xi} = \frac{\eta'}{\xi'} \approx \frac{\xi'}{\eta'};$$

so that $(\xi')^2 = (\eta')^2$ as well.

The interpretation of a path in (ξ, η) space is slightly more subtle than usual. In (x, y, z) space, any z coordinate can be assigned to any (x, y) pair. Transformation into (ξ, η) space means that any (ξ, η) is consistent with any non-zero and finite z (the case of $z \in \{0, \pm\infty\}$ is addressed below). Thus, each path in (ξ, η) space represents an infinite number of paths in (x, y, z) space, the paths being indexed by non-zero and finite z. The same (ξ, η) point may therefore represent both $(\xi(z), \eta(z))$ and $(\xi(-z), \eta(-z))$ for any non-zero and finite z. When $z < 0$, though, the flowlines are taken to indicate decreasing values of z.[5]

The following lemmata reveal that only a small number of points in (ξ, η) space are consistent with $z \in \{0, \pm\infty\}$. As admissible $x(z)$ and $y(z)$ must be defined for all $z \in \Re$, it is necessary that candidate FNE strategies contain these points.

Lemma 5.15 *Candidate FNE solutions in (ξ, η) space must contain an element of $\mathcal{S}$.*

Proof. Admissible strategies must be defined for $z = 0$. It is shown that the only (ξ, η) points consistent with $z = 0$ are those in $\mathcal{S}$.

When $z = 0$ but $x, y \neq 0$, the change of variables defined in equations 5.19 implies $(\xi, \eta) = \mathbf{0}$, an element of $\mathcal{S}$. When $z = x = 0$ but $y \neq 0$, L'Hôpital's rule and the first order condition in equation 5.10 imply that $\xi = \frac{1}{x'} = \frac{y}{y+z} = 1$ while $\eta = 0$, as above. This produces $(\xi, \eta) = (1, 0)$, another element of $\mathcal{S}$. Symmetry of ξ and η produces the third element of $\mathcal{S}$ when $y = z = 0$ but $x \neq 0$.

When $x = y = z = 0$ L'Hôpital's rule again provides $\xi = \frac{1}{x'}$ and $\eta = \frac{1}{y'}$. But as $x = y$ it must be that $\xi = \eta$ so that $x' = y'$ as well. Therefore, first order conditions 5.9 and 5.10 become $x' = 1 + \frac{1}{x'}$ so that $x' = a$. Hence

[5] Replacing z with $-z$ such that $\xi(z) = \xi(-z)$ and $\eta(z) = \eta(-z)$ changes the signs in equations 5.20 but not in equation 5.21.

$(\xi, \eta) = \left(\frac{1}{a}, \frac{1}{a}\right)$ which, by definition of a, produces the final two elements of $\mathcal{S}$, $(-a, -a)$. ■

Similarly, by considering those points consistent with infinite z:

Lemma 5.16 *Candidate FNE solutions in (ξ, η) space must satisfy one of the following conditions:*

1. *either one of ξ or η becomes infinite as z does; or*
2. *$(\xi, \eta) \to (-a, -a)$ as z becomes infinite.*

Proof. Admissible strategies must be defined for $z = \pm\infty$. For any finite x and y the change of variables in equations 5.19 means that infinite z implies infinite ξ and η.

When x is finite but y infinite again $\xi \to \pm\infty$. Now, though, $\eta = \frac{z}{y}$ so that L'Hôpital's rule is invoked for

$$\eta = \frac{1}{y'} = \left(1 + \frac{z}{x}\right)^{-1} = \frac{1}{1+\xi} = 0.$$

A similar result holds when x is infinite but y finite.

The preceding established the first condition of the lemma; the second is now established. When x and y are both infinite, L'Hôpital's rule is again used. As above

$$\eta = \frac{1}{1+\xi} \text{ and } \xi = \frac{1}{1+\eta};$$

so that $\eta + \xi\eta = \xi + \xi\eta = 1$. Therefore $\xi = \eta$ and $1 - \xi - \xi^2 = 0$ so that $(\xi, \eta) = (-a, -a)$. ■

These lemmata shed some light on the two linear solutions to Problem 5.7 found by Lemma 5.9: the points $(\xi, \eta) = (-a, -a)$ are the only points in (ξ, η) space that are consistent with all $z \in \Re$. Furthermore, if z is finite and non-zero at either of these points, $\xi' = \eta' = 0$ and the 'path' necessarily remains at the point.

These lemmata identify various families of paths that are defined for all $z \in \Re$. As candidate strategies must also avoid regions in (ξ, η) space that violate the second order conditions (the second and fourth quadrants and the third above $1 - \xi - \eta = 0$), FNE strategies may only be taken from the following families:

$\mathcal{D}_1$ the two linear FNE;

$\mathcal{D}_2$ the continuum of paths converging on $(-a_-, -a_-)$ from above;

$\mathcal{D}_3$ the continuum of paths converging on $(-a_-, -a_-)$ from the origin;

$\mathcal{D}_4$ the two paths converging on $(-a_-, -a_-)$ from $(0, 1)$ and $(1, 0)$, respectively; and

$\mathcal{D}_5$ the two paths diverging asymmetrically from $(-a_+, -a_+)$.

The following lemmata eliminate all but the $\mathcal{D}_1$ family from consideration.

Lemma 5.17 *None of the $\mathcal{D}_2$ paths support FNE.*

Proof. These paths cannot be parameterised by $z \in \Re$. Suppose, for example, that $z = -\infty$ at $(\xi, \eta) = (\infty, \infty)$. The flowlines prevent (ξ, η) returning to $(-a_-, -a_-)$ as z increases to zero. The reverse also holds: if $z = \infty$ at $(\xi, \eta) = (\infty, \infty)$ the flowlines again prevent (ξ, η) returning to $(-a_-, -a_-)$ as z increases to zero. ∎

The next two lemmata use a similar argument. They find that the paths considered either behave as above or become discontinuous (and are therefore not solutions to the original differential equations).

Lemma 5.18 *None of the $\mathcal{D}_3$ paths support FNE.*

Proof. As $(\xi, \eta) = \mathbf{0}$ is consistent with $z = 0$, but not $z = \pm\infty$, the path considered contains the points $(\xi(-\infty), \eta(-\infty)) = (-a_-, -a_-)$, $(\xi(0), \eta(0)) = \mathbf{0}$ and $(\xi(\infty), \eta(\infty)) = (-a_-, -a_-)$. By Lemma 5.15, both $x(0)$ and $y(0)$ are non-zero. In (ξ, η) space, $(\xi(\hat{z}), \eta(\hat{z})) = (\xi(-\hat{z}), \eta(-\hat{z}))$ for any given $\hat{x}$. Therefore, by the change of variables, $\frac{z}{x(z)} = \xi(z) = \xi(-z) = \frac{-z}{x(-z)}$ so that $x(z) = -x(-z)$. As $x(0) \neq 0$ (and $y(0) \neq 0$) this is discontinuous at $z = 0$ and therefore does not satisfy equations 5.9 and 5.10, which are continuous at $z = 0$. ∎

Lemma 5.13 and Figure 5.1, which addressed the case of symmetric play, illustrate more clearly the argument above: $x(z) = -x(-z)$ implies a symmetric solution jumping from one path to its mirror image in the horizontal axis at $z = 0$. Those paths continuous at $x(0)$ may also be seen in Figure 5.1.

Lemma 5.19 *Neither of the $\mathcal{D}_4$ paths support FNE.*

Proof. The proof here differs from that of Lemma 5.18 in one respect: $x(0)$ is still distinct from zero, but $y(0) = 0$. L'Hôpital's rule, though, shows that $x'(0) = 1 + \frac{1}{y'(0)}$. As $y'(0) = 1 + \frac{z}{x(0)} = \frac{x(0)+z}{x(0)} = 1$, $x'(0) = 2$. As $x'(0)$ is well defined, the solution must still be continuous. The $\mathcal{D}_4$ paths are not. ∎

Lemma 5.20 *Neither of the $\mathcal{D}_5$ paths support FNE.*

Proof. Consider $\mathcal{D}_5$ paths at $\xi = \eta$. Here $x = y = z = 0$ so that the ODEs may be written

$$x'(0) = 1 + \frac{1}{y'(0)} \text{ and } y'(0) = 1 + \frac{1}{x'(0)}.$$

Multiplying both by their denominators yields $x'(0)\, y'(0) = 1 + x'(0) = 1 + y'(0)$ so that $x'(0) = y'(0)$, a symmetric solution. As the $\mathcal{D}_5$ paths are not symmetric, they do not satisfy the original differential equations. ■

The $x' = y'$ that does satisfy the above is a, as expected.

Having eliminated all candidates:

Theorem 5.21 *There are no non-linear FNE of Problem 5.7 when $\delta = 0$.*

Symmetric linear solutions to this problem are more efficient than are asymmetric linear solutions. When play is symmetric, the agents split the task, and quadratic cost, of offsetting the ε shocks which cause z to deviate from zero. While this is true, efficiency arguments do not necessarily provide insight into game play.

One of the interesting features of the theorem is that it holds for any non-degenerate distribution of ε. This contrasts to the finding in Klemperer and Meyer [KM89]. There the support of ε influences the set of FNE by altering the zone over which second order conditions must hold.

5.4 The Pareto frontier

Before asking whether transfers increase expected utility, the Pareto frontier is traced for Problem 5.7. Weight objective functions 5.4 and 5.5 by ϕ and $(1-\phi)$, respectively, so that social welfare is

$$\begin{aligned} w &= \phi u + (1-\phi)\, v \\ &= -\phi x^2 - \phi\,(z-\delta)^2 - (1-\phi)\, y^2 - (1-\phi)\,(z+\delta)^2 . \end{aligned}$$

As the planner maximises social welfare subject to equation 5.1, the state variable, it must

$$\begin{aligned} & \max_{x,y} E\,[w] \\ = \; & \max_{x,y} \int_{-\bar{\varepsilon}}^{\bar{\varepsilon}} \left[-\phi\left(2x^2 + y^2 + \delta^2 + \varepsilon^2 + 2xy + 2\varepsilon\,(x+y) - 2\delta\,(x+y+\xi)\right) \right. \\ & \left. -(1-\phi)\left(x^2 + \delta^2 + \varepsilon^2 + 2\,(y+\varepsilon)\,(x+y) + 2\delta\,(x+y+\varepsilon)\right)\right] f(\varepsilon)\, d\varepsilon . \end{aligned}$$

Differentiating with respect to z, as above, is possible as the social planner controls $z(\varepsilon)$. This, though, only produces a single first order condition. Therefore continue, instead, to regard the choice variables as x and y. As the gradients in these must be zero for all realisations of ε, necessary conditions are derived by differentiating the square bracketed expression to obtain the Euler conditions.

Differentiating with respect to x produces

$$\begin{aligned}(\phi+1)\,x^{*}+y+\varepsilon+(1-2\phi)\,\delta &= 0;\\ x^{*} &= \frac{2\phi-1}{\phi}\delta-\frac{1}{\phi}z\\ &= \frac{1}{\phi}\left[(2\phi-1)\,\delta-z\right].\end{aligned}$$

Similarly, differentiation with respect to y yields

$$y^{*}=\frac{1}{1-\phi}\left[(2\phi-1)\,\delta-z\right].$$

By equation 5.1, then,

$$\begin{aligned}z &= \frac{(2\phi-1)\,\delta-z}{\phi\,(1-\phi)}+\varepsilon;\\ z &= \frac{(2\phi-1)\,\delta+\phi\,(1-\phi)\,\varepsilon}{\phi\,(1-\phi)+1}.\end{aligned}$$

This allows the closed form solutions

$$\begin{aligned}x^{*} &= \frac{(1-\phi)\,(2\phi-1)\,\delta-(1-\phi)\,\varepsilon}{\phi\,(1-\phi)+1};\\ y^{*} &= \frac{\phi\,(2\phi-1)\,\delta-\phi\varepsilon}{\phi\,(1-\phi)+1}.\end{aligned}$$

Substitution using the assumption that $E[\varepsilon]=0$ produces the individual payoffs:

$$E[u] = -\frac{(1-\phi)^{2}\left(1+\phi^{2}\right)}{\left[\phi\,(1-\phi)+1\right]^{2}}\left[5\delta^{2}+E\left[\varepsilon^{2}\right]\right];\text{ and} \tag{5.22}$$

$$E[v] = -\frac{\phi^{2}\left(1+(1-\phi)^{2}\right)}{\left[\phi\,(1-\phi)+1\right]^{2}}\left[5\delta^{2}+E\left[\varepsilon^{2}\right]\right]. \tag{5.23}$$

Together equations 5.22 and 5.23 define a curve parameterised by ϕ. In $(E[u],E[v])$ space, the curve is concave and symmetric about $E[u]=E[v]$. Furthermore, it is tangent to the horizontal axis ($E[v]=0$) at $\phi=0$ and to the vertical axis ($E[u]=0$) at $\phi=1$.

In the case of symmetric agents ($\delta=0,\phi=\frac{1}{2}$), expected payoffs simplify to

$$E[u]=E[v]=-\frac{1}{5}E\left[\varepsilon^{2}\right]. \tag{5.24}$$

As the problem has reduced to the concave

$$\min_{z(\varepsilon)}\int_{-\bar{\varepsilon}}^{\bar{\varepsilon}}\left[\frac{5}{4}z^{2}-\frac{1}{2}\varepsilon z+\frac{1}{4}\varepsilon^{2}\right]f(\varepsilon)\,d\varepsilon;$$

the first order conditions are sufficient.

5.4.1 Comparison with the game

As expected, the payoffs to the planned outcome in equation 5.24 are greater than those to the game (presented in Theorem 5.11).

The case of symmetric agents ($\delta = 0, \phi = \frac{1}{2}$) also allows comparison of the responsiveness of the optimal controls, $\frac{\partial x}{\partial \varepsilon}$. In the game, $x = za$ and $z = \frac{\varepsilon}{1-2a}$ so that $\frac{\partial x}{\partial \varepsilon} = \frac{a}{1-2a} = \left\{-\frac{1+\sqrt{5}}{2\sqrt{5}}, \frac{1-\sqrt{5}}{2\sqrt{5}}\right\}$. In the planned environment, $x = -2z$ and $z = \frac{1}{5}\varepsilon$. Therefore $\frac{\partial x}{\partial \varepsilon} = -\frac{2}{5}$, intermediate to those of the game.

In all cases, $\frac{\partial x}{\partial \varepsilon} < 0$, which has a sensible interpretation: in the symmetric case, both agents wish to set their controls to zero and would like the state to be zero. Without shocks, this could be achieved; therefore shocks are to be counteracted. That the response of the planned outcome is intermediate suggests that the game responses reflect the failure of the game to attain the planned outcome.

5.5 FNE with transfers

Now augment agents' (symmetric) objective functions to define a new problem:

Problem 5.22 (transfer) *Agents 1 and 2 have objective functions*

$$\begin{aligned} u &= -x(z)^2 - z^2 - r(y) + s(x); \\ v &= -y(z)^2 - z^2 + r(y) - s(x); \end{aligned}$$

where $x(z)$ and $y(z)$ are as above but $r(y) \geq 0$ is now a transfer controlled by agent 1 and $s(x) \geq 0$ one controlled by agent 2.

This formulation is therefore one of transferable utility. Clearly this does not correspond as closely to the motivating problem as would a model of transfers in consumption goods. The formulation is adopted, though, for analytical tractability. Nevertheless, x and y are referred to as emission functions and r and s as transfer functions. It is not clear what effect such linear transfers have on the existence of FNE.

Extending the previous approach, so that agent 1 chooses an $x(z)$ and a $r()$ to maximise $E[u]$ against 2's fixed $y(z)$ and $s()$, no longer suffices: a fixed $y^*(z)$ only responds to changes in z. If z is not influenced by transfers then neither are emissions. Two approaches therefore present themselves as possible. The first involves specifying $x(r-s, z)$ and $y(r-s, z)$. Making emissions a function of the transfers is appealing: it regards the quantity possessed of the transferable resource as a second payoff relevant state variable. Unfortunately, this approach involves PDEs. For this reason it is not pursued.

The second involves considering a three stage game in fictional time in which agents first announce and commit to their transfer functions. They next calculate and commit to optimal $x()$ and $y()$; finally, ε is realised. This is, of course, an *ad hoc* formulation, converting the game into a form of Stackelberg problem. This formulation also raises questions about the commitment technology assumed. Nevertheless, this approach is pursued as an initial attempt at the problem.

First, though, what transfers are required to support the first best?

5.5.1 Implementing the symmetric first best

The calculations in Section 5.4 show that the first best is supported by emissions of the form $x^* = y^* = -2z$ when agents are symmetric. As these are not a FNE of the game without transfers, it is now asked what transfers are capable of supporting these emission functions. The transfer functions are not here required to support an equilibrium.

In the second stage, agents regard the transfers $r(y)^*, s(x)^*$ and the other's x or y as fixed. Agent 1 therefore must

$$\max_z E\left[-x(z)^2 - z^2 - r^*(y^*(z)) + s^*(x(z))\right] \text{ s.t. } \begin{array}{r} z = x(z) + y^*(z) + \varepsilon \\ y^* = -2z; \end{array}$$

or

$$\max_z E\left[-(3z-\varepsilon)^2 - z^2 - r^*(-2z) + s^*(3z-\varepsilon)\right].$$

This has first order conditions[6]

$$-6(3z-\varepsilon) - 2z + 2r^{*\prime}(-2z) + 3s^{*\prime}(3z-\varepsilon) = 0.$$

Implementing the first best for symmetric agents requires that $x^* = y^* = -2z$ and $r^* = s^*$. This allows simplification of the above to

$$\begin{aligned} 5r^{*\prime}(y) &= 2z + 6y \\ &= 5y; \end{aligned}$$

so that

$$r^*(y) = \frac{1}{2}y^2 + c. \tag{5.25}$$

The non-negativity requirement will be satisfied for any $c \geq 0$. As $c > 0$ simply imposes an additional cost on the transferring agent without altering the recipient's incentives, assume that

$$c = 0. \tag{5.26}$$

[6] As $r(y), s(x) \geq 0$ these may be non-differentiable at certain points. The solutions that are eventually derived are differentiable for all z.

This is a counter-intuitive form for a transfer function to have, paying out as the other agent's emissions deviate from the ideal of $y = 0$.

Expected utility may now be calculated. The functional forms set $z = \frac{1}{5}\varepsilon$ so that $x^* = y^* = -\frac{2}{5}\varepsilon$ and $r^* = s^* = -\frac{2}{25}\varepsilon^2$. Therefore

$$\begin{aligned} E[u] &= E[v] = -\frac{4}{25}E\left[\varepsilon^2\right] - \frac{1}{25}E\left[\varepsilon^2\right] - \frac{2}{25}E\left[\varepsilon^2\right] + \frac{2}{25}E\left[\varepsilon^2\right] \\ &= -\frac{1}{5}E\left[\varepsilon^2\right]; \end{aligned}$$

the same as the planned outcome. The transfer function in equation 5.25 therefore must maximise, rather than minimise, the objective functions.

5.5.2 FNE in the three stage game

It is naturally of interest to ask whether the above emissions and transfer scheme is an equilibrium. Noting that it has linear emissions and quadratic transfer functions, functions are restricted to these classes in the following section. More general transfer functions are considered after that. These sections find that, when functions are so restricted, there is an FNE with linear emissions and quadratic transfers (although not the first best above). The final section shows that, when emissions are constrained to be linear, the optimal transfers are quadratic but conjectures that the linear-quadratic result does not generally hold when both functions are not constrained.

Linear emissions and quadratic transfers

Equilibria in linear emissions functions, $x(z) = \xi z$ and $y(z) = \eta z$, and quadratic transfer functions, $r(y) = \rho y^2$ and $s(x) = \sigma x^2$ (non-negativity requires that $\rho, \sigma \geq 0$) are now sought. This is clearly restrictive.

The second stage In the second stage, agents must

$$\begin{aligned} &\max_x \left\{ E\left[-x^2 - z^2 - \rho^*(y^*)^2 + \sigma^* x^2\right] \text{ s.t. } z = x + y^* + \varepsilon \right\}; \\ &\max_y \left\{ E\left[-y^2 - z^2 + \rho^* y^2 - \sigma^*(x^*)^2\right] \text{ s.t. } z = x^* + y + \varepsilon \right\}; \end{aligned}$$

or

$$\begin{aligned} &\max_z E\left[(\sigma^* - 1)(z - y^* - \varepsilon)^2 - z^2 - \rho^*(y^*)^2\right]; \\ &\max_z E\left[(\rho^* - 1)(z - x^* - \varepsilon)^2 - z^2 - \sigma^*(x^*)^2\right]. \end{aligned}$$

These problems have first order conditions

$$\begin{aligned} y'(z)\left[(\sigma - 1)x(z) + \rho y(z)\right] &= (\sigma - 1)x(z) - z; \\ x'(z)\left[(\rho - 1)y(z) + \sigma x(z)\right] &= (\rho - 1)y(z) - z. \end{aligned}$$

As x and y are restricted to linear strategies, these necessary conditions simplify to

$$\begin{aligned} \rho\eta^2 + (\sigma - 1)\,\xi\eta - (\sigma - 1)\,\xi + 1 &= 0; \text{ and} \quad (5.27) \\ \sigma\xi^2 + (\rho - 1)\,\xi\eta - (\rho - 1)\,\eta + 1 &= 0; \quad (5.28) \end{aligned}$$

when $z \neq 0$. Unfortunately, equations 5.27 and 5.28 produce lengthy expressions for $\xi(\rho, \sigma)$ and $\eta(\rho, \sigma)$ that are not amenable to interpretation. Note that $(\xi = \eta = -2) \Rightarrow \left(\rho = \sigma = \frac{1}{2}\right)$, the first best discovered in Section 5.5.1.

The second order conditions require that

$$\begin{aligned} (\sigma^* - 1)\left[\left(1 - y^{*\prime}\right)^2 - (z - y^* - \varepsilon)\,y^{*\prime\prime}\right] - 1 - \rho^*\left[\left(y^{*\prime}\right)^2 + y^* y^{*\prime\prime}\right] &\leq 0; \text{ and} \\ (\rho^* - 1)\left[\left(1 - x^{*\prime}\right)^2 - (z - x^* - \varepsilon)\,x^{*\prime\prime}\right] - 1 - \sigma^*\left[\left(x^{*\prime}\right)^2 + x^* x^{*\prime\prime}\right] &\leq 0; \end{aligned}$$

respectively. The functional form restrictions adopted here reduce these to

$$\begin{aligned} (\sigma - 1)\,(1 - \eta)^2 - 1 - \rho\eta^2 &\leq 0; \text{ and} \quad (5.29) \\ (\rho - 1)\,(1 - \xi)^2 - 1 - \sigma\xi^2 &\leq 0. \quad (5.30) \end{aligned}$$

The first stage Consider now the problem facing agents in the first stage. Let $\xi(\rho, \sigma)$ and $\eta(\rho, \sigma)$ be candidate best response correspondences, therefore solving equations 5.27 and 5.28. Given these, it is necessary that $\frac{\partial E[u]}{\partial \rho}|_{\sigma^*} = \frac{\partial E[v]}{\partial \sigma}|_{\rho^*} = 0$. Working with agent 1,

$$E[u] = E\left[-\xi^2(\rho, \sigma)\,z^2 - z^2 - \rho\eta^2(\rho, \sigma)\,z^2 + \sigma\xi^2(\rho, \sigma)\,z^2\right], \quad (5.31)$$

where

$$z = \frac{\varepsilon}{1 - \xi(\rho, \sigma) - \eta(\rho, \sigma)}.$$

Differentiating with respect to ρ and assuming that $1 - \xi - \eta \neq 0$ then produces a necessary condition for the first stage,

$$\begin{aligned} 0 = (1 - \xi - \eta)&\left[-2\xi\xi_\rho - \eta^2 - 2\rho\eta\eta_\rho + 2\sigma\xi\xi_\rho\right] \\ &+2\,(\xi_\rho + \eta_\rho)\left[(\sigma - 1)\,\xi^2 - 1 - \rho\eta^2\right]. \quad (5.32) \end{aligned}$$

The equivalent expression for agent 2 is

$$\begin{aligned} 0 = (1 - \xi - \eta)&\left[-2\eta\eta_\sigma - \xi^2 + 2\rho\eta\eta_\sigma - 2\sigma\xi\xi_\sigma\right] \\ &+2\,(\xi_\sigma + \eta_\sigma)\left[(\rho - 1)\,\eta^2 - 1 - \sigma\xi^2\right]. \quad (5.33) \end{aligned}$$

Partial differentiation of equations 5.27 and 5.28 with respect to ρ and

σ produces

$$\begin{bmatrix} 2\rho\eta + (\sigma-1)\xi & (\sigma-1)(\eta-1) & 0 & 0 \\ 0 & 0 & 2\rho\eta + (\sigma-1)\xi & (\sigma-1)(\eta-1) \\ (\sigma-1)(\xi-1) & 2\sigma\xi + (\rho-1)\eta & 0 & 0 \\ 0 & 0 & (\sigma-1)(\xi-1) & 2\sigma\xi + (\rho-1)\eta \end{bmatrix}$$
$$\times \begin{bmatrix} \eta_\rho & \xi_\rho & \eta_\sigma & \xi_\sigma \end{bmatrix}'$$
$$= \begin{bmatrix} -\eta^2 & \xi(1-\eta) & \eta(1-\xi) & -\xi^2 \end{bmatrix}';$$

so that, when the 4×4 matrix is invertible,

$$\Delta \begin{bmatrix} \eta_\rho & \xi_\rho & \eta_\sigma & \xi_\sigma \end{bmatrix}' \tag{5.34}$$
$$= - \begin{bmatrix} \eta\left[(\sigma+1)\xi\eta + (\rho-1)\eta^2 + (\sigma-1)(\xi+\eta-1)\right] \\ (\xi-1)\left[(\rho+1)\eta^2 + (\sigma-1)\xi\eta\right] \\ (\eta-1)\left[(\sigma+1)\xi^2 + (\rho-1)\xi\eta\right] \\ \xi\left[(\rho+1)\xi\eta + (\sigma-1)\xi^2 + (\rho-1)(\xi+\eta-1)\right] \end{bmatrix};$$

where

$$\Delta \equiv 2\left[(\rho\eta+\sigma\xi)^2 - \rho\eta^2 - \sigma\xi^2\right] + (\rho\sigma - \rho - \sigma + 1)(\xi+\eta-1).$$

Agent 1's second order conditions now require that

$$\begin{aligned} \frac{\partial^2 E[u]}{\partial\rho^2} &= E\left\{2\left[z_\rho^2 + zz_{\rho\rho}\right]\left[(\sigma-1)\xi^2(\rho,\sigma) - 1 - \rho\eta^2(\rho,\sigma)\right]\right. \\ &\quad +2zz_{\rho\rho}\left[2(\sigma-1)\xi\xi_\rho - 2\rho\eta\eta_\rho\right] \\ &\quad +2zz_{\rho\rho}\left[-2\xi\xi_\rho - \eta^2 - 2\rho\eta\eta_\rho + 2\sigma\xi\xi_\rho\right] \\ &\quad \left. +2z^2\left[\sigma\left(\xi_\rho^2 + \xi\xi_{\rho\rho}\right) - \left(\xi_\rho^2 + \xi\xi_{\rho\rho}\right) - 2\eta\eta_\rho - \rho\left(\eta_\rho^2 + \eta\eta_{\rho\rho}\right)\right]\right\} \\ &\leq 0; \end{aligned}$$

where

$$\begin{aligned} z_\rho &= \varepsilon(\xi_\rho + \eta_\rho)\left[1 - \xi(\rho,\sigma) - \eta(\rho,\sigma)\right]^{-2}; \\ z_{\rho\rho} &= \varepsilon\left\{(\xi_{\rho\rho} + \eta_{\rho\rho})\left[1 - \xi(\rho,\sigma) - \eta(\rho,\sigma)\right]^{-2}\right. \\ &\quad \left. +2(\xi_\rho + \eta_\rho)^2\left[1 - \xi(\rho,\sigma) - \eta(\rho,\sigma)\right]^{-3}\right\}; \\ \xi_{\rho\rho} &= \frac{\xi_\rho^2}{1-\xi} + \frac{(1-\xi)\left[\eta^2 + 2(\rho+1)\eta\eta_\rho + (\sigma-1)(\xi_\rho\eta + \xi\eta_\rho)\right]}{\Delta(\rho,\sigma)} \\ &\quad -\frac{\xi_\rho}{\Delta(\rho,\sigma)}\Delta_\rho; \\ \eta_{\rho\rho} &= \frac{\eta_\rho\Delta_\rho}{\Delta} - \frac{\eta_\rho^2}{\eta} \\ &\quad -\frac{\eta\left[(\sigma+1)\xi_\rho\eta + \xi\eta_\rho + \eta^2 + 2(\rho-1)\eta\eta_\rho + (\sigma-1)(\xi_\rho + \eta_\rho)\right]}{\Delta}; \\ \Delta_\rho &= 2\left[2(\rho\eta + \sigma\xi)(\eta + \rho\eta_\rho + \sigma\xi_\rho) - \eta^2 - 2\rho\eta\eta_\rho - 2\sigma\xi\xi_\rho\right] \\ &\quad +(\sigma-1)(\xi+\eta-1) + (\rho\sigma - \rho - \sigma + 1)(\xi_\rho + \eta_\rho). \end{aligned}$$

As $E\left[\varepsilon^2\right] \geq 0$, this reduces to

$$\begin{aligned}&\left[3\left(\xi_\rho+\eta_\rho\right)^2+\left(\xi_{\rho\rho}+\eta_{\rho\rho}\right)\left[1-\xi-\eta\right]\right]\left[\left(\sigma-1\right)\xi^2-1-\rho\eta^2\right]\\&+\left\{\left(\xi_{\rho\rho}+\eta_{\rho\rho}\right)\left[1-\xi-\eta\right]+2\left(\xi_\rho+\eta_\rho\right)^2\right\}\left[4\left(\sigma-1\right)\xi\xi_\rho-4\rho\eta\eta_\rho-\eta^2\right]\\&+\left[1-\xi-\eta\right]^2\left[\left(\sigma-1\right)\left(\xi_\rho^2+\xi\xi_{\rho\rho}\right)-2\eta\eta_\rho-\rho\left(\eta_\rho^2+\eta\eta_{\rho\rho}\right)\right]\leq 0;\end{aligned} \tag{5.35}$$

when $1-\xi-\eta \neq 0$.

Symmetric conditions exist for agent 2's problem.

Lemma 5.23 *The symmetric first best, as identified in Section 5.5.1, does not form an FNE of Problem 5.22.*

Proof. Substitution of $\rho=\sigma=\frac{1}{2}$ and $\xi=\eta=-2$ into equations 5.27, 5.28, 5.32 and 5.33 solves the first two but not the second two. ■

Symmetric solutions Consider situations in which $\sigma=\rho$ and $\eta=\xi$. In these cases the system of equations 5.27, 5.28, 5.32 and 5.33 can be reduced to two equations. The first order conditions of the second stage reduce to

$$\left(2\rho-1\right)\xi^2-\left(\rho-1\right)\xi+1=0. \tag{5.36}$$

The first order conditions for the first stage become

$$\begin{aligned}&2\left[\left(2\rho-1\right)\xi^2-\left(\rho-1\right)\xi+1\right]\xi_\rho\\&+2\left[\left(1-2\rho\right)\xi^2+\rho\xi+1\right]\eta_\rho+\left(1-2\xi\right)\xi^2=0.\end{aligned}$$

As the first term of this is identically zero when equation 5.36 is satisfied, this reduces further to

$$2\left[\left(1-2\rho\right)\xi^2+\rho\xi+1\right]\eta_\rho+\left(1-2\xi\right)\xi^2=0. \tag{5.37}$$

Finally, the expression for η_ρ is derived from matrix 5.34:

$$\eta_\rho=-\frac{\xi\left[2\rho\xi^2+\left(2\xi-1\right)\left(\rho-1\right)\right]}{4\rho\xi^2\left(2\rho-1\right)+\left(\rho-1\right)^2\left(2\xi-1\right)}. \tag{5.38}$$

A solution to equations 5.36 and 5.37 is obtained by manipulation. First rearrange equation 5.36 for

$$\rho\xi\left(2\xi-1\right)=\xi^2-\xi-1; \tag{5.39}$$

which, in turn, may be rearranged for

$$\begin{aligned}\left(\rho-1\right)\xi\left(2\xi-1\right)+\xi^2+1&=0; &(5.40)\\ \left(2\rho-1\right)\xi\left(2\xi-1\right)+\xi^2+1-\rho\xi\left(2\xi-1\right)&=0.\end{aligned}$$

Equation 5.39 then simplifies this to

$$(2\rho - 1)\,\xi\,(2\xi - 1) + \xi + 2 = 0. \tag{5.41}$$

Finally, the square bracketed term in equation 5.37 may be rearranged using equation 5.39 for

$$2\,[2 + \xi]\,\eta_\rho = (2\xi - 1)\,\xi^2. \tag{5.42}$$

Equation 5.38 may be divided by $2\xi - 1$ as this cannot be a solution (substitution of it into equation 5.36 produces $\frac{5}{4} = 0$); this produces

$$\begin{aligned}
\frac{\eta_\rho}{2\xi - 1} &= -\frac{\left[2\rho\xi^2 + (2\xi - 1)(\rho - 1)\right]\xi}{4\rho\xi\,(2\xi - 1)(2\rho - 1)\,\xi + \left[(\rho - 1)(2\xi - 1)\right]^2} \\
&= \frac{2\rho\xi^3 + (2\xi - 1)\,\xi\,(2\rho - 1) - \rho\xi\,(2\xi - 1)}{4\rho\xi\,(\xi + 2) - \left[(\xi^2 + 1)\,\xi^{-1}\right]^2};
\end{aligned}$$

where the numerator has been expanded and the denominator simplified by equations 5.41 and 5.40, respectively. Now simplify the numerator by equation 5.41 for

$$\frac{\eta_\rho}{2\xi - 1} = \frac{2\rho\xi^3 - \xi - 2 - \rho\xi\,(2\xi - 1)}{4\rho\xi\,(\xi + 2) - (\xi^2 + 1)^2\,\xi^{-2}};$$

and again by equation 5.39 for

$$\begin{aligned}
\frac{\eta_\rho}{2\xi - 1} &= \frac{2\rho\xi^3 - \left(\xi^2 + 1\right)}{4\rho\xi\,(\xi + 2) - (\xi^2 + 1)^2\,\xi^{-2}} \\
&= \frac{2\rho\xi^3 - \left(\xi^2 + 1\right)}{4\rho\xi^3 - \frac{(\xi^2+1)^2}{\xi+2}}\frac{\xi^2}{\xi + 2}.
\end{aligned}$$

By equation 5.42, though,

$$\frac{\eta_\rho}{2\xi - 1} = \frac{\frac{1}{2}\xi^2}{2 + \xi};$$

so that the above becomes

$$\frac{2\rho\xi^3 - \left(\xi^2 + 1\right)}{4\rho\xi^3 - \frac{(\xi^2+1)^2}{\xi+2}}\frac{\xi^2}{\xi + 2} = \frac{\frac{1}{2}\xi^2}{2 + \xi}.$$

Removing the common factors leaves

$$\begin{aligned}
2\rho\xi^3 - \left(\xi^2 + 1\right) &= \frac{1}{2}\left[4\rho\xi^3 - \frac{\left(\xi^2 + 1\right)^2}{\xi + 2}\right]; \\
2\left(\xi^2 + 1\right) &= \frac{\left(\xi^2 + 1\right)^2}{\xi + 2}.
\end{aligned}$$

Pareto rank	description	$E[u] = E[v]$	$E[z^2]$
1	first best	$-\frac{1}{5}E[\varepsilon^2]$	$\frac{1}{25}E[\varepsilon^2]$
2	transfer game, $x'(z) < 0$	$-\frac{2}{9}E[\varepsilon^2]$	$\frac{1}{9}E[\varepsilon^2]$
3	no transfer game, $x'(z) < 0$	$-\frac{5-\sqrt{5}}{10}E[\varepsilon^2]$	$\frac{1}{5}E[\varepsilon^2]$
4	transfer game, $x'(z) > 0$	$-\frac{2}{5}E[\varepsilon^2]$	$\frac{1}{25}E[\varepsilon^2]$
5	no transfer game, $x'(z) > 0$	$-\frac{5+\sqrt{5}}{10}E[\varepsilon^2]$	$\frac{1}{5}E[\varepsilon^2]$

Table 5.1: Expected utility for symmetric agents

As $\xi^2 \neq -1$, this leaves a quadratic. Thus

$$(\rho, \xi) = \left\{ \left(\frac{1}{3}, -1\right), \left(\frac{1}{3}, 3\right) \right\}; \tag{5.43}$$

satisfy the first order conditions.

Both roots satisfy the second order conditions of the second stage, as identified by equations 5.29 and 5.30. As for those of the first stage, some tedious algebra determines that $(\rho, \sigma, \xi, \eta) = \left(\frac{1}{3}, \frac{1}{3}, -1, -1\right)$ sets equation 5.35 to $-\frac{1269}{16}$ while $(\rho, \sigma, \xi, \eta) = \left(\frac{1}{3}, \frac{1}{3}, 3, 3\right)$ sets it to $-\frac{218565}{16}$. Thus both real solutions are maxima. As $\rho = \sigma > 0$ in both cases, transfers are always non-negative. Hence:

Theorem 5.24 *The elements of equation 5.43 form a FNE of Problem 5.22 when emissions are constrained to be linear, and transfers quadratic.*

Equation 5.31 allows calculation of expected payoffs. For $(\rho, \sigma, \xi, \eta) = \left(\frac{1}{3}, \frac{1}{3}, -1, -1\right)$ these are $E[u] = E[v] = -\frac{2}{9}E[\varepsilon^2]$ while for $(\rho, \sigma, \xi, \eta) = \left(\frac{1}{3}, \frac{1}{3}, 3, 3\right)$ they are $E[u] = E[v] = -\frac{2}{5}E[\varepsilon^2]$. Thus the two FNE are Pareto ranked. Table 5.1 compares these expected payoffs to those arising from the game without transfers and from the social planner's solution.

As expected, the social planner's outcome Pareto dominates. Otherwise, while each equilibrium in the game with transfers is preferred to its counterpart in the game without transfers, agents' preferences between the games will depend on the probabilities assigned to each of the two FNE by an equilibrium selection mechanism.

One of the peculiar results in Table 5.1 is that expected variance of z for the FNE that sets x as a positive function of z is as low as that of the first best, and much less than its Pareto preferred alternative. This also stands in contrast to the expected variance of z in the game without transfers; in that case, both FNE yield the same result.

Asymmetric solutions Return to the general case in which symmetry may not hold by using NAG's hybrid Powell method root finder for non-linear systems, `c05tbc`. This is used to simultaneously solve equations 5.27, 5.28, 5.32 and 5.33. Initial $(\rho, \sigma, \xi, \eta)$ seeds are randomly drawn from a uniform distribution over $(-10, 10)$. Most seeds fail to make progress between iterations, or exceed 200 iterations without finding roots. Even repeating the programme until roots have been found successfully from 400 different seeds only identifies the two symmetric real roots found in equation 5.43. This suggests that there may not be asymmetric real equilibria to problem 5.22.

Against this optimistic belief, Bézout's Theorem suggests that, in general, the number of complex intersections is the product of the degrees of the algebraic curves. In this case, as equations 5.27 and 5.28 are third order in ρ, σ, ξ and η and equations 5.32 and 5.33 seem to be of eleventh order, the product is 1089. It is somewhat unsettling, given this potentially large number of roots, that no asymmetric equilibria have been found.

Linear emissions but general transfers

In this section, emission functions continue to be restricted to linear functions, but general transfers, $r(y), s(x) \geq 0$, are now allowed.

The second stage Without first imposing the non-negativity constraints, the problems at the second stage are to

$$\max_z E\left[-(z-y-\varepsilon)^2 - z^2 - r^*(y) + s^*(z-y-\varepsilon)\right]; \quad (5.44)$$

$$\max_z E\left[-(z-x-\varepsilon)^2 - z^2 + r^*(z-x-\varepsilon) - s^*(x)\right]. \quad (5.45)$$

These have first order conditions

$$\begin{aligned} -2x\left(1-y'\right) - 2z - r'y' + s'\left(1-y'\right) &= 0; \\ -2y\left(1-x'\right) - 2z + r'\left(1-x'\right) - s'x' &= 0. \end{aligned}$$

If $x(z)$ and $y(z)$ are linear functions, then it is necessary that

$$\begin{bmatrix} -\eta & 1-\eta \\ 1-\xi & -\xi \end{bmatrix} \begin{bmatrix} r'(y) \\ s'(x) \end{bmatrix} = 2z \begin{bmatrix} \xi(1-\eta)+1 \\ \eta(1-\xi)+1 \end{bmatrix};$$

so that

$$\begin{bmatrix} r'(y) \\ s'(x) \end{bmatrix} = \frac{2z}{(1-\xi-\eta)} \begin{bmatrix} \xi^2 - \xi^2\eta - \eta^2 + \xi\eta^2 - \xi\eta + \xi + 1 \\ -\xi^2 + \xi^2\eta + \eta^2 - \xi\eta^2 - \xi\eta + \eta + 1 \end{bmatrix};$$

when $1-\xi-\eta \neq 0$. To integrate substitute out z for

$$\begin{bmatrix} r'(y) \\ s'(x) \end{bmatrix} = \begin{bmatrix} 2\frac{\xi^2-\xi^2\eta-\eta^2+\xi\eta^2-\xi\eta+\xi+1}{(1-\xi-\eta)\eta} y \\ 2\frac{-\xi^2+\xi^2\eta+\eta^2-\xi\eta^2-\xi\eta+\eta+1}{(1-\xi-\eta)\xi} x \end{bmatrix};$$

so that

$$\begin{bmatrix} r(y) \\ s(x) \end{bmatrix} = \begin{bmatrix} \frac{\xi^2-\xi^2\eta-\eta^2+\xi\eta^2-\xi\eta+\xi+1}{(1-\xi-\eta)\eta} y^2 + c_y \\ \frac{-\xi^2+\xi^2\eta+\eta^2-\xi\eta^2-\xi\eta+\eta+1}{(1-\xi-\eta)\xi} x^2 + c_x \end{bmatrix};$$

where c_x, c_y are constants of integration.

One sufficient condition that satisfies the non-negativity requirement on transfers is that $c_x, c_y = \infty$. As this is unpalatable, it could be assumed that $c_x, c_y = 0$, in keeping with the reasoning behind equation 5.26 that previously set the constant to zero. In general, non-negativity would then require determination of which regions in (ξ, η) space satisfy it. But this is unnecessary: setting $c_x, c_y = 0$, returns the transfers to quadratic forms, allowing application of the previous results. It is easily seen the ρ, σ, ξ, η combinations presented in equation 5.43 are consistent with this more general formulation.

Therefore:

Theorem 5.25 *Given linear emissions functions, continuous transfer functions capable of forming FNE to Problem 5.22 must be quadratic.*

The linear-quadratic FNE in the broader functional space

This section does not constrain emission and transfer functions to be linear or quadratic, as above. Instead, it merely requires that they be members of $\mathcal{C}^1$. The principal question asked here is whether the FNE derived above remain so in this broader functional space.

Lemma 5.26 *Agent 1's optimal emission function against*

$$\begin{aligned} y(z) &= -z; \\ s(x) &= \frac{1}{3}x^2; \text{ and} \\ r(y) &= \frac{1}{3}y^2; \end{aligned}$$

in Problem 5.22 is $x(z) = -z$.

Proof. Agent 1's problem is to

$$\max_x E\left[-x^2 - z^2 - r(y) + s(x)\right];$$

subject to the assumptions above and the state equation 5.1. These restrictions reduce the problem to

$$\max_z E\left[-4z^2 + \frac{8}{3}z\varepsilon - \frac{2}{3}\varepsilon^2\right];$$

which has first order conditions setting

$$z = \frac{1}{3}\varepsilon.$$

By state equation 5.1 and $y = -z$,

$$x^* = -z.$$

■

The same technique determines that agent the optimal emission function against $y(z) = 3z$, $s(x) = \frac{1}{3}x^2$ and $r(y) = \frac{1}{3}y^2$ is $x(z) = 3z$. Thus, both of the FNE in the restricted function space remain FNE.

It is more difficult to confirm that the quadratic transfers are optimal when not constrained. The technique just used in the lemma cannot be modified so that $x(z)$ is fixed but $r(y)$ is not as, given fixed emissions, agents will not transfer. Previously, when functional forms were constrained, the second stage calculations allowed derivation of best responses $\xi(\rho, \sigma)$ and $\eta(\rho, \sigma)$. These, substituted into the first stage, then allowed a problem in ρ and σ. This approach was possible as the assumption of quadratic transfer functions allowed these functions to be identified by a single parameter, something that cannot be done in general. As a result, only a conjecture is presented here:

Conjecture 5.27 *When admissible strategy functions in Problem 5.22 are members of $\mathcal{C}^1$, strategies supporting a FNE depend on the distribution of the stochastic variable, ε. Thus, linear emissions and quadratic transfers do not generally support a FNE.*

This conjecture is supported by initial calculations in which the agents' problems are reduced to optimal control problems of the standard forms in which the co-state equations depend on the distribution of ε.

5.6 Discussion

This chapter has presented a number of results for specific models. First, models displaying both existence and non-existence of FNE were presented. In the first model with an FNE, a result different from that in in Klemperer and Meyer [KM89] was found. In their linear model, increasing the support of the stochastic variable decreases the size of the equilibrium set in their work, by requiring the second order conditions to hold over a larger domain. Here, the results simply rely on a non-degenerate support for the stochastic variable.

Thirdly, a social planner can implement the first best (linear) emissions by designing quadratic transfers. While these transfers do not necessarily support non-cooperative equilibria, they did provide an initial guess that equilibrium transfers might also be quadratic. FNE were found when functions were restricted to be linear (for emissions) and quadratic (for transfers), but it is not expected that these results hold when the functions are less constrained.

The constraints on the functions in the game with transfers make the resulting FNE difficult to compare to those in the game without transfers. The comparison presented in Table 5.1 suggests that, while transfers may allow Pareto improvements even when they net out in equilibrium, they do not necessarily do so. It is unclear whether broadening the function spaces from which emission and transfer functions can be selected will Pareto improve or worsen the resulting FNE: if the game is regarded principally as a control problem then the broader control space can only be of benefit; if, though, it is regarded as a cooperation problem, then the extra freedom may give agents another means of engaging in strategic behaviour.

This chapter also leaves open some further questions. For example, are there non-linear FNE in the quadratic model without transfers? While the differential games examined previously are clearly different to the one-shot games examined here, they do suggest that non-linear strategies may be unusual in games with linear state equations are quadratic objective functions.

Is it possible to make general statements about when additional control variables allow Pareto improvements? On the surface, this question resembles somewhat that of when adding an additional instrument in an incomplete market is Pareto improving. An obvious difference, of course, is that one of these situations is a game while the other is a market; more generally, it is not clear whether this parallel has analytical importance.

Finally, the present agents are defined in a symmetric fashion, differing only by $\pm\delta$. To the extent that FNE are of interest, they are likely to be of interest when players may differ in other ways as well.

Chapter 6

Conclusions

This thesis set out to examine the question of how greenhouse gas emissions might evolve in the absence of an enforceable mechanism for international cooperation. This complicated question has been addressed by exploring two more tractable questions. The first involved examination of a linear-quadratic differential game, the second one-shot games in which emissions and transfers were specified as functions.

Improved analysis of non-linear strategies in the linear-quadratic game has provided two new results. First, the 1990 continuum result of Tsutsui and Mino [TM90] for symmetric play by symmetric agents is qualified: the non-linear continuum, previously thought to survive under all conditions, is seen only to survive under certain calibrations. No meaningful economic intuition could be found to explain the conditions under which the continuum survives.

This qualification was achieved by noting that admissible strategies must allow agents to calculate the consequences of playing all admissible deviations from proposed equilibrium play. This recognition was important in analysis of both questions addressed by this thesis.

As Tsutsui and Mino's continuum result had been used as the basis for other results with non-linear strategies, the present qualification hopefully provides a firmer basis for future developments in this area. As differential games provide a natural generalisation of both optimal control problems and single-shot games, this firmer footing may be very useful.

The second set of results derive from numerical analysis of asymmetric play and asymmetric agents. This suggests that no asymmetric non-linear strategies exist. At present, the evidence for this is merely strong rather than conclusive. The interpretation of this finding seems simply to be that the symmetric case, as a degenerate case, has properties not shared by asymmetric cases. From a computational point of view, the uniqueness of linear MPE in linear-quadratic games is desirable: its calculation is considerably simpler than that of non-linear strategies.

Obviously, the class of linear-quadratic games is very restricted within the larger world of differential games. There is therefore no reason to believe that any of the results on linear-quadratic games carry over to the more general setting. However, the numerical tools developed in this thesis can be applied to these more general settings by changing the differential equations being solved.

There is clearly room for further development of differential games' use in the study of non-cooperative greenhouse gas emissions. The numerical techniques used here allow more plausible functional forms and richer interactions than those used here (for reasons of tractability and benchmarking) to be specified without much difficulty. More difficult to incorporate, but equally plausible, would be time-varying characteristics. Similarly, there are at least two sources of uncertainty in the practical problem: understanding of how the climate system responds to anthropogenic emissions is still quite limited; and knowledge about the relationship between climate and national welfare is lacking.

An interest in transfers led to analysis of functional Nash equilibria in the one-shot game. This has provided some additional insight into conditions under which models may fail to have FNE. Second, and perhaps more importantly, a model has been presented in which uncertainty's role in refining the equilibrium set is more dramatic than it is in Klemperer and Meyer's study [KM89]. Third, it has been demonstrated that the addition of a second instrument can be Pareto improving. This result still deserves further elaboration as the games with and without transfers cannot yet be Pareto ranked.

To the extent that the one-shot game is regarded as a special case of the earlier differential game, a fourth observation may also be made, weakening that claim: while the differential games yielded a single linear solution, the functional games studied yield two linear solutions, one upward sloping and one downward. The latter is more consistent with the finding in the differential game but, in the absence of any further selection mechanism, has nothing to argue for it over the upward sloping strategy. As in the differential game, it is possible that these multiple equilibria are a feature of symmetric agents and play.

Returning to the thesis' motivating question, some extensions to the existing work suggest themselves. First, calibration is ultimately necessary to address this practical question. There is, though, a trade-off between calibrating a simple model and trying to develop a more sophisticated model. The possibilities for the latter are certainly plentiful. Strategies, for example, have been assumed continuous throughout this thesis. While this may often be plausible, it would be preferable were continuity derived rather than assumed. One approach to discontinuous strategies makes use of Skiba's Theorem, whereby control discontinuities occur in such a way as to maintain value function continuity [Ski78]. Jensen and Lockwood present conditions sufficient to rule out the possibility of discontinuous strategies in affine-quadratic differential games [JL98]. These, at it turns out, are identical to the sufficient conditions for unique affine MPE strategies presented in Lockwood [Loc96], suggesting that the differential games examined here do not admit discontinuous strategies.

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